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THE STRAIGHT LINE SOLUTIONS OF THE PROBLEM OF *N* BODIES*

By F. R. Moulton

1. Introduction. In an elegant prize memoir,† Lagrange proved that there are, for any three finite masses attracting one another according to the Newtonian law, four distinct configurations such that, under proper initial projections, the *ratios of the mutual distances remain constants.* In all the configurations the bodies describe similar conic sections with respect to the center of mass of the system, the simplest case being that in which the orbits are circular. In three of the four solutions the masses lie always in a straight line, and in the fourth they remain at the vertices of an equilateral triangle.

In this paper two closely related problems are treated:

I. The number of straight line solutions is found for n arbitrary positive masses‡; that is, the ratios of the distances are determined so that under proper initial projections the bodies will always remain collinear. This is the direct generalization to the problem of n bodies of Lagrange's straight line solutions. The method of Lagrange is not suitable for treating the general case. Below the problem is formulated so that it is solved by finding the total number of real solutions of n simultaneous fractional algebraic equations. From the mathematical standpoint the interest in this part of the paper centers in the algebraical problem of finding the number of real solutions of a formidable looking set of simultaneous equations.

II. The second problem is that of determining, when possible, n masses such that if they are placed at n arbitrary collinear points, they will, under proper initial projection, always remain in a straight line. If n is even and the linear dimensions of the orbits are given, it is proved that the n masses are in general uniquely determined; and that if n is odd the coördinates of the n points must satisfy one algebraic relation, after which, choosing any one of the

*The greater part of this paper was written in 1900 and was presented to the Chicago Section of the Am. Math Soc., Dec. 28, 1900. See *Bull. Am. Math. Soc.*, vol. 7 (1900–1901), p. 249. It has been recast and put in form for publication by the author as a Research Associate of the Carnegie Institution of Washington.

† Lagrange, *Collected Works*, vol. 6, pp. 229–324.

‡ Treated briefly by Lehmann-Filhès in *Astronomische Nachrichten*, vol. 127 (1891), no. 033.

masses arbitrarily, the remaining $n-1$ are uniquely determined. This problem is in a certain sense the converse of the preceding.

I. Determination of the Positions when the Masses are given.

2. The Equations Defining the Solutions. Let the origin of coördinates be taken at the center of gravity of the system. This point and the line of initial projection of any mass determine a plane. All the other masses must be projected in this plane, for otherwise they would not be collinear at the end of the first element of time. All the bodies being initially in a line and projected in the same plane, they will always remain in this plane. Consequently if solutions exist in which the n masses are always in a straight line, the orbits are plane curves.

Let the plane of motion be the $\xi\eta$ — plane. Let the masses be denoted by $m_1, m_2, \cdots, m_n$, and their respective coördinates by (ξ_1, η_1), (ξ_2, η_2), $\cdots$, (ξ_n, η_n). Then, choosing the units so that the Gaussian constant is unity, the differential equations of motion are

$$
(1) \quad \left\{
\begin{aligned}
&\frac{d^2\xi_i}{dt^2} = \frac{1}{m_i}\frac{\partial U}{\partial \xi_i}, \quad (i = 1, \cdots, n), \\
&\frac{d^2\eta_i}{dt^2} = \frac{1}{m_i}\frac{\partial U}{\partial \eta_i}, \\
&U = \tfrac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{m_j\, m_k}{r_{jk}}, \quad (j \neq k), \\
&r_{jk} = \sqrt{(\xi_j - \xi_k)^2 + (\eta_j - \eta_k)^2}.
\end{aligned}
\right.
$$

Equations (1) admit the integral of areas

$$
(2) \quad \left\{
\begin{aligned}
&\sum_{i=1}^{n} m_i \left(\xi_i \frac{d\eta_i}{dt} - \eta_i \frac{d\xi_i}{dt}\right) = \sum_{i=1}^{n} m_i r_i^2 \frac{d\theta_i}{dt} = c, \\
&r_i = \sqrt{\xi_i^2 + \eta_i^2}.
\end{aligned}
\right.
$$

In case the n bodies remain collinear the line of the resultant acceleration to which each one is subject always passes through the origin. Therefore

in collinear solutions we have, by the law of areas, for each body separately

$$m_i r_i^2 \frac{d\theta_i}{dt} = c_i, \quad (i = 1, \cdots, n).$$

But when the bodies remain collinear we have also

$$\frac{d\theta_1}{dt} = \frac{d\theta_2}{dt} = \cdots = \frac{d\theta_n}{dt},$$

from which it follows that

$$\text{(3)} \qquad \frac{r_i}{r_j} = \sqrt{\frac{m_j c_i}{m_i c_j}} = a_{ij},$$

where the a_{ij} are constants. That is, if any collinear solutions exist the ratios of the distances of the bodies from the origin are constants, and it easily follows from this that the ratios of their mutual distances are also constants. They are therefore of the Lagrangian type.

If the n masses remain collinear, the ratios of their coördinates are constants, being equal to the ratios of their distances from the origin. Therefore in all collinear solutions we have

$$\text{(4)} \qquad \xi_i = x_i \xi, \quad \eta_i = x_i \eta, \quad (i = 1, \cdots, n),$$

where the x_i are constants. Substituting in equations (1) we have, as necessary conditions for the existence of the collinear solutions,

$$\text{(5)} \qquad \begin{cases} \dfrac{d^2\xi}{dt^2} = -\displaystyle\sum_{j=1}^{n} \frac{m_j(x_i - x_j)}{x_i[(x_i - x_j)^2]^{3/2}} \frac{\xi}{r^3}, \quad j \neq i, \ (i = 1, \cdots, n), \\[2ex] \dfrac{d^2\eta}{dt^2} = -\displaystyle\sum_{j=1}^{n} \frac{m_j(x_i - x_j)}{x_i[(x_i - x_j)^2]^{3/2}} \frac{\eta}{r^3}, \\[2ex] r = \sqrt{\xi^2 + \eta^2}. \end{cases}$$

In order that ξ and η as defined by their initial values and equations (5) shall be the same for all values of i, the coefficients of ξ/r^3 and η/r^3 must be set equal to a constant independent of i. Letting $-\omega^2$ represent this constant and $r_{ij} = \sqrt{(x_i - x_j)^2}$, these conditions, which are sufficient as well as necessary for the existence of the collinear solutions, become

$$
(6)\quad\begin{cases}
0 + \dfrac{m_2(x_1 - x_2)}{r_{12}^3} + \dfrac{m_3(x_1 - x_3)}{r_{13}^3} + \cdots + \dfrac{m_n(x_1 - x_n)}{r_{1n}^3} = \omega^2 x_1,\\[2ex]
\dfrac{m_1(x_2 - x_1)}{r_{21}^3} + 0 + \dfrac{m_3(x_2 - x_3)}{r_{23}^3} + \cdots + \dfrac{m_n(x_2 - x_n)}{r_{2n}^3} = \omega^2 x_2,\\
\cdot \qquad\qquad \cdot \qquad\qquad \cdot\\
\cdot \qquad\qquad \cdot \qquad\qquad \cdot\\
\cdot \qquad\qquad \cdot \qquad\qquad \cdot\\
\dfrac{m_1(x_n - x_1)}{r_{n1}^3} + \dfrac{m_2(x_n - x_2)}{r_{n2}^3} + \dfrac{m_3(x_n - x_3)}{r_{n3}^3} + \cdots + 0 = \omega^2 x_n.
\end{cases}
$$

It will be shown that ω^2 must be positive in order that real solutions of these equations shall exist. Suppose the notation is chosen so that in any solution $x_1 < \cdots < x_n$. Then the terms of the left member of the last equation are all positive. Therefore $\omega^2 > 0$ unless $x_n < 0$, which is impossible since the center of gravity relation $m_1 \xi_1 + \cdots + m_n \xi_n = 0$ holds true.

For every set of real values of $x_1, \cdots, x_n$ satisfying equations (6) the solutions of (5) are the same for all i, and these solutions substituted in (4) give the coördinates in the collinear configurations.

Since equations (5) have the same form as the differential equations in the two-body problem, it follows that *in the collinear solutions the orbits are similar conic sections.* In case the orbits are ellipses the coefficient of $-\xi/r^3$ and $-\eta/r^3$ is the product of the cube of the major semi-axis of the orbit and the square of the mean angular speed of revolution. If the undetermined scale factor be chosen so that x_i is the major semi-axis of the orbit of m_i, then the mean angular velocity of revolution of the system is ω.

The hypothesis is made that ω^2 and $m_1, \cdots, m_n$ are real positive numbers, and the problem is to find the number of real solutions of (6) for any value of n. For each of these solutions there is a six-fold infinity of collinear configurations, the six arbitrary parameters being the two which define the plane of motion, the one which defines the orientation of the orbits in their plane, the one which determines the epochs at which the bodies pass their apses, the one which determines the scale of the system, and finally the eccentricity of the orbits.

3. Outline of the Method of Solution. The solution involves a mathematical induction and consists of the following steps:

Assumption (A). To get the induction, it is assumed that for $n = \nu$ the number of real solutions of (6) for $x_1, \cdots, x_\nu$ is N_ν, whatever real positive values ω^2 and $m_1, \cdots, m_\nu$ may have. It is known from the work of Lagrange that when $\nu = 3$ we have $N_3 = 3 = \frac{1}{2}\, 3\,!$.

Theorem (B). If to the system $m_1, \cdots, m_\nu$ of positive masses an infinitesimal mass * $m_{\nu+1}$ be added, then the whole number of real solutions is

$$(\nu + 1)\, N_\nu.$$

Theorem (C). As the infinitesimal mass $m_{\nu+1}$ increases continuously to any finite positive value whatever, the total number of real solutions remains precisely $(\nu + 1)\, N_\nu$.

Conclusion (D). By successive applications of theorems (B) and (C) it is seen that the number of real solutions of (6) for $n + \nu + \mu$ is

$$N_{\nu+\mu} = (\nu + \mu)(\nu + \mu - 1) \cdots (\nu + 2)(\nu + 1)\, N_\nu.$$

When $\nu = 3$ it is known that $N_3 = \frac{1}{2} 3\,!$. Therefore

$$N_{3+\mu} = \tfrac{1}{2}(\mu + 3)\,!.$$

Let $\mu + 3 = n$ and we have

$$N_n = \tfrac{1}{2}\, n\,!. \tag{7}$$

To complete the demonstration of this conclusion it remains only to prove theorems (B) and (C).

4. Proof of Theorem (B). When there are ν finite bodies $m_1, \cdots, m_\nu$ and the infinitesimal body $m_{\nu+1}$, equations (6) become

---

* The "infinitesimal body" is in celestial mechanics a well established definite thing. In the first place it is absolutely zero. But obviously if this were all it would have no interest. It is used in association with certain differential equations which are derived in writing them first for all the masses real and positive, then letting one of them (the infinitesimal mass) approach zero. The limit of this variable mass, zero, is the infinitesimal mass, and as this mass approaches zero, its motion satisfies the differential equations, which at the limit zero for the variable mass define the motion of the infinitesimal body. Since in this limiting process the differential equations remain determinate and in all respects regular the process is fully justified. It is precisely in this sense that the infinitesimal mass is used in Theorem (B); the equations used are the limits of equations for all masses finite as $m_{\nu+1}$ approaches 0. In Theorem (C) the mass $m_{\nu+1}$ increases continuously and we follow the roots of the algebraic equations, at least as to their character.

$$
(8)\quad \begin{cases}
\phi_1 \equiv -\omega^2 x_1 + 0 + \dfrac{m_2(x_1 - x_2)}{r_{12}^3} + \cdots + \dfrac{m_\nu(x_1 - x_\nu)}{r_{1\nu}^3} + \dfrac{m_{\nu+1}(x_1 - x_{\nu+1})}{r_{3\,\nu+1}} = 0, \\
\phi_2 \equiv -\omega^2 x_2 + \dfrac{m_1(x_2 - x_1)}{r_{12}^3} + 0 + \cdots + \dfrac{m_\nu(x_2 - x_\nu)}{r_{2\nu}^3} + \dfrac{m_{\nu+1}(x_2 - x_{\nu+1})}{r_{2\,\nu+1}^3} = 0, \\
\cdot \\
\cdot \\
\cdot \\
\phi_\nu \equiv -\omega^2 x_\nu + \dfrac{m_1(x_\nu - x_1)}{r_{1\nu}^3} + \dfrac{m_2(x_\nu - x_2)}{r_{2\nu}^3} + \cdots + 0 + \dfrac{m_{\nu+1}(x_\nu - x_{\nu+1})}{r_{\nu\,\nu+1}^3} = 0, \\
\phi_{\nu+1} \equiv -\omega^2 x_{\nu+1} + \dfrac{m_1(x_{\nu+1} - x_1)}{r_{1\,\nu+1}^3} + \dfrac{m_2(x_{\nu+1} - x_2)}{r_{2\,\nu+1}^3} + \cdots + \dfrac{m_\nu(x_{\nu+1} - x_\nu)}{r_{\nu\,\nu+1}^3} + 0 = 0.
\end{cases}
$$

The last column of these equations is zero because $m_{\nu+1} = 0$, but is written for use in the proof of Theorem (C). Consequently the first ν equations, which involve $x_1, \cdots, x_\nu$ alone as unknowns, are the equations defining the solutions when $n = \nu$. By (A) there are N_ν real solutions of these equations. Let any one of these solutions be $x_1 = x_1^{(o)}, \cdots, x_\nu = x_\nu^{(o)}$. Then the last equation of (8) becomes

$$
(9)\quad \phi_{\nu+1} \equiv -\omega^2 x_{\nu+1} + \frac{m_1(x_{\nu+1} - x_1^{(o)})}{r_{1\,\nu+1}^3} + \frac{m_2(x_{\nu+1} - x_2^{(o)})}{r_{2\,\nu+1}^3} + \cdots + \frac{m_\nu(x_{\nu+1} - x_\nu^{(o)})}{r_{\nu\,\nu+1}^3} = 0.
$$

The number of real solutions of this equation is required.

Consider $\phi_{\nu+1}$ as a function of $x_{\nu+1}$. It is easily verified that

$$
(10)\quad \begin{cases}
\phi_{\nu+1}(+\infty) = -\infty, \\
\lim\limits_{\epsilon=0} \phi_{\nu+1}(x_j^{(o)} + \epsilon) = +\infty, \quad (j = 1, \cdots, \nu), \\
\lim\limits_{\epsilon=0} \phi_{\nu+1}(x_j^{(o)} - \epsilon) = -\infty, \quad (j = 1, \cdots, \nu), \\
\phi_{\nu+1}(-\infty) = +\infty.
\end{cases}
$$

Since $\phi_{\nu+1}$ is finite and continuous except at $x_{\nu+1} = x_1^{(o)}, \cdots, x_{\nu+1} = x_\nu^{(o)}$, $+\infty$, $-\infty$, it follows that there is an odd number of real solutions in each of the intervals $-\infty$ to $x_p^{(o)}$, where $x_p^{(o)}$ is the smallest $x_j^{(o)}$; $x_k^{(o)}$ to $x_l^{(o)}$, where $x_k^{(o)}$ and $x_l^{(o)}$ are any two $x_j^{(o)}$ which are adjacent; and $x_q^{(o)}$ to $+\infty$, where $x_q^{(o)}$ is the largest $x_j^{(o)}$.

But we find from (9) that

$$\frac{\partial \phi_{\nu+1}}{\partial x_{\nu+1}} \equiv -\omega^2 - \frac{2\,m_1}{r_{1\,\nu+1}^3} - \frac{2\,m_2}{r_{2\,\nu+1}^3} - \cdots - \frac{2\,m_\nu}{r_{\nu\,\nu+1}^3},$$

which is negative except at $x_{\nu+1} = x_1, \cdots, x_{\nu+1} = x_\nu$, where it is infinite. Therefore $\phi_{\nu+1}$ is a decreasing monotonic function in each of the intervals, and consequently vanishes once, and but once, in each of them. Since there are $\nu + 1$ of these intervals there are, for each real solution of the first ν equations of (8), precisely $\nu + 1$ real solutions of the last equation of (8). Since the first ν equations have, by (A), N_ν real solutions, equations (8) altogether have precisely $(\nu + 1)\ N_\nu$ real solutions. This completes the demonstration of Theorem (B).

4. Proof of Theorem (C). Let $x_j = x_j^{(0)}$ $(j = 1, \cdots, \nu + 1)$ be any one of the $(\nu + 1)\,N_\nu$ real solutions of equations (8) which are known to exist for $m_{\nu+1} = 0$. It will be shown that as $m_{\nu+1}$ increases continuously to any finite positive quantity whatever, the $x_j^{(0)}$ may be made to vary continuously so as always to satisfy equations (8), and that during these variations the $x_j^{(0)}$ remain distinct, finite, and real. From this it will follow that there are at least $(\nu + 1)\ N_\nu$ real solutions of (8) for every set of finite positive values of $m_1, \cdots, m_{\nu+1}$. It will also be shown that no new solutions can appear as $m_{\nu+1}$ increases from zero to any finite value. Hence it will follow that the number of real solutions of (8) is exactly $(\nu + 1)\ N_\nu$ for all finite positive values of $m_1, \cdots, m_{\nu+1}$.

The roots of algebraic equations are continuous functions of the coefficients so long as the roots are finite and the equations do not have indeterminate forms. Consequently the $x_j^{(0)}$ are continuous functions of $m_{\nu+1}$ if no $x_i^{(0)}$ becomes infinite and if no $x_i^{(0)} = x_j^{(0)}$. The real roots of algebraic equations having real coefficients can disappear only by passing to infinity, or by an even number of real solutions becoming conjugate complex quantities in pairs. Therefore we have to determine (1) whether any finite $x_i^{(0)}$ can become equal to any $x_j^{(0)}$, (2) whether any $x_i^{(0)}$ can become infinite, and (3) whether any two real solutions can become conjugate complex quantities for any finite positive values of $m_1, \cdots, m_{\nu+1}$.

(1). The masses $m_1, \cdots, m_{\nu+1}$ are by hypothesis all positive. Let the notation be chosen so that for any values of $m_1, \cdots, m_{\nu+1}$ for which the $x_i^{(0)}$ are all distinct we have the inequalities $x_1^{(0)} < x_2^{(0)} < \cdots < x_\nu^{(0)} < x_{\nu+1}^{(0)}$. Let us suppose that as some mass is changed we find $x_j^{(0)} - x_i^{(0)}$ approaching zero in such a way that $x_j^{(0)}$ and $x_i^{(0)}$ remain finite. That is,

r_{ij}, which occurs only in the expressions for ϕ_i and ϕ_j, approaches zero. Suppose $i < j$. Then the term involving r_{ij} becomes negatively infinite in ϕ_i and positively infinite in ϕ_j. Consider $\phi_i = 0$. Another r_{ik} must approach zero in order to restore the finite value of the function ϕ_i, and the term involving r_{ik} must become positively infinite as r_{ik} approaches zero. Therefore we see from (8) that $k < i$. But r_{ik} enters besides only in ϕ_k, and similar reasoning shows that r_{kl}, where $l < k$, must also approach zero. In this manner we are driven to the conclusion finally that an r_{pq}, where one of the subscripts is unity, approaches zero. Then consider $\phi_1 = 0$. All its terms except $-\omega^2 x_1$ are negative, and since one of its r_{ij}, that is r_{pq}, approaches zero, the first equation of (8) can not be satisfied. Consequently the original assumption that some r_{ij} can approach zero for finite values of $x_1^{(0)}, \cdots, x_{\nu+1}^{(0)}$ and finite *positive* values of $m_1, \cdots, m_{\nu+1}$ leads to an impossibility, and is therefore false.

(2). Multiplying equations (8) by $m_1, m_2, \cdots, m_{\nu+1}$ respectively and adding, we get

$$-\omega^2(m_1x_1 + m_2x_2 + \cdots + m_\nu x_\nu + m_{\nu+1}\, x_{\nu+1}) = 0.$$

It follows from this equation that no $x_i^{(0)}$ alone can become infinite, and that if one becomes positively infinite, then some other one must become negatively infinite.

Suppose the notation is again chosen so that $x_1^{(0)} < x_2^{(0)} < \cdots < x_\nu^{(0)} < x_{\nu+1}^{(0)}$. Then if any $x_i^{(0)}$ becomes negatively infinite $x_1^{(0)}$ must do the same, and from the equation above it follows that $x_{\nu+1}^{(0)}$ must become positively infinite. Now suppose this to occur and consider the equation $\phi_1 = 0$. In order that this equation may remain satisfied, $x_2^{(0)}$ must also become negatively infinite in such a way that $x_1^{(0)} - x_2^{(0)}$ shall approach zero. But now it follows from $\phi_2 = 0$, since $-\omega^2 x_2^{(0)}$ and $\dfrac{m_1\,(x_2^{(0)} - x_1^{(0)})}{r_{12}^3}$ are both positive, that $x_3^{(0)}$ must also become negatively infinite in such a way that $x_2^{(0)} - x_3^{(0)}$ shall approach zero. It follows similarly from $\phi_3 = 0$ that $x_4^{(0)}$ must become negatively infinite in such a way that $x_3^{(0)} - x_4^{(0)}$ shall approach zero. This reasoning continues until it is found that $x_1^{(0)}, \cdots, x_{\nu+1}^{(0)}$ must all become negatively infinite. But $x_{\nu+1}^{(0)}$ at least must become positively infinite, and we are thus led to a contradiction. Similarly it can be proved that no $x_i^{(0)}$ can become positively infinite.

In order to prove now that as $m_{\nu+1}$ approaches zero, equations (8)

remain determinate and their solutions finite, and that there are accordingly no others besides those obtained in Theorem (B), consider a solution $x_1, \cdots, x_{\nu+1}$ in which the x_j will be distinct for a set of positive values of $m_1, \cdots, m_{\nu+1}$, and then let m_i approach zero as a limit.

In the first place, if x_i approaches neither x_{i-1} nor x_{i+1} as a finite limit as m_i approaches zero as a limit, then by the reasoning of (1) and (2) above no x_j can approach any x_k as a limit.

In the second place, x_i can not approach x_{i+1} as a finite limit as m_i approaches zero unless x_{i-1} also approaches x_{i+1}, for otherwise $\phi_i = 0$ can not be satisfied. But if x_{i-1} approaches x_{i+1} as a limit as m_i approaches zero, then $\phi_{i-1} = 0$ and $\phi_{i+1} = 0$ can not be satisfied unless x_{i-2} and x_{i+2} also approach x_{i+1} as a limit. This shifts the difficulty to $\phi_{i-2} = 0$ and $\phi_{i+2} = 0$, and so on until $\phi_1 = 0$ and $\phi_{\nu+1} = 0$ are reached, which can not be satisfied under the hypothesis.

In the third place, x_i can not become positively infinite as m_i approaches zero, for then $\phi_i = 0$ can not be satisfied unless x_{i-1} becomes infinite in such a way that $x_i - x_{i-1}$ approaches zero. Continuing through $\phi_{i-1} = 0$, etc., we are led to the conclusion that $x_1, \cdots, x_{\nu+1}$ all become positively infinite; but then the center of gravity equation can not be satisfied. In a similar manner it can be proved that as m_i approaches zero as a limit no x_j can approach any x_k or become infinite. Consequently the solutions all remain regular as m_i approaches zero as a limit.

(3). Since the solutions of (8) are continuous functions of $m_{\nu+1}$, it follows that no two solutions which are real for $m_{\nu+1} = 0$ can ever become conjugate complex solutions for any real value of $m_{\nu+1}$ without having first become equal; and similarly, no two solutions which are complex for $m_{\nu+1} = 0$ can ever become real for any real value of $m_{\nu+1}$ without having first become equal. Consequently if a multiple solution of (8) is impossible for every set of finite positive values of $m_1, \cdots, m_{\nu+1}$, it is impossible that any real solutions should disappear by becoming imaginary, or that any imaginary solutions should become real.

The conditions that $x = x^{(0)}$ shall be a multiple solution of $f(x) = 0$ are that $f(x^{(0)}) = 0$ and $f'(x^{(0)}) = 0$. The corresponding conditions that a set of simultaneous algebraic equations shall have a multiple solution are that a set of values of the variables shall satisfy the equations and that the Jacobian of the functions with respect to the dependent variables shall vanish for the same set of values. That is, the conditions that $x_j = x_j^{(0)}$,

$(j = 1, \cdots, \nu + 1)$, shall be a multiple solution of (8) are that these values shall satisfy (8) and also the equation

$$(11) \qquad \Delta \equiv \begin{vmatrix} \dfrac{\partial \phi_1}{\partial x_1} & \dfrac{\partial \phi_1}{\partial x_2} & \cdots & \dfrac{\partial \phi_1}{\partial x_{\nu+1}} \\ \dfrac{\partial \phi_2}{\partial x_1} & \dfrac{\partial \phi_2}{\partial x_2} & \cdots & \dfrac{\partial \phi_2}{\partial x_{\nu+1}} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \dfrac{\partial \phi_{\nu+1}}{\partial x_1} & \dfrac{\partial \phi_{\nu+1}}{\partial x_2} & \cdots & \dfrac{\partial \phi_{\nu+1}}{\partial x_{\nu+1}} \end{vmatrix} = 0.$$

Consider two solutions of a set of algebraic equations having real coefficients. If they change from real to conjugate complex quantities or from conjugate complex to real quantities for a certain value of a continuously varying parameter, then for this value of the parameter, they are not only equal, but they are also *real*. Consequently we shall have occasion to examine Δ only when all of its elements are real. We shall show that it cannot vanish for any finite set of real values of the x_j when $m_1, \cdots, m_{\nu+1}$ are positive, and consequently that it can not vanish for any particular set which satisfies equations (8). When this is established we shall have proved that all the solutions of (8) which are real for $m_{\nu+1} = 0$ remain real when $m_{\nu+1}$ increases to any positive value, and those which are complex remain complex.

From equations (8) and (11) we find

$$(12) \qquad \Delta \equiv \begin{vmatrix} a_1 & \dfrac{2m_2}{r_{12}^3} & \cdots & \dfrac{2m_{\nu+1}}{r_{1\,\nu+1}^3} \\ \dfrac{2m_1}{r_{21}^3} & a_2 & \cdots & \dfrac{2m_{\nu+1}}{r_{2\,\nu+1}^3} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \dfrac{2m_1}{r_{\nu+1\,1}^3} & \dfrac{2m_2}{r_{\nu+1\,2}^3} & \cdots & a_{\nu+1} \end{vmatrix}$$

where

$$a_1 = -\omega^2 - 0 - \frac{2m_2}{r_{12}^3} - \cdots - \frac{2m_{\nu+1}}{r_{1\nu+1}^3},$$

$$a_2 = -\omega^2 - \frac{2m_1}{r_{21}^3} - 0 - \frac{2m_3}{r_{23}^3} - \cdots - \frac{2m_{\nu+1}}{r_{2\nu+1}^3},$$

$$\vdots$$

$$a_{\nu+1} = -\omega^2 - \frac{2m_1}{r_{\nu+11}^3} - \cdots - \frac{2m_\nu}{r_{\nu+1\nu}^3} - 0.$$

If $m_{\nu+1} = 0$ this determinant breaks up into the product of a determinant of the same type as (12) when all the m_i are distinct from zero and a factor essentially negative. Therefore it is sufficient to consider the general case in which all the m_i are positive in examining whether Δ can vanish or not.

Several properties of Δ are evident. (a) If the ith row be multiplied by m_i ($i = 1, \cdots, \nu + 1$), the determinant becomes symmetrical. (b) The sum of the elements in each row is $-\omega^2$, from which it follows that the expansion of the determinant contains ω^2 as a factor. (c) The expansion of the determinant contains $(-1)^{\nu+1}\omega^{2(\nu+1)}$ as one of its terms, and since all the m_i are positive and all the x_i are real the sign of all the terms coming from the product of the elements of the main diagonal is $(-1)^{\nu+1}$.

When Δ is completely expanded all those terms not having the sign $(-1)^{\nu+1}$ are cancelled by terms coming from the product of the main diagonal elements, and since the term $(-1)^{\nu+1}\omega^{2(\nu+1)}$ is certainly present the determinant can in no case be zero. The following demonstration of this fact was invented in 1907 by Mr. T. H. Hildebrandt, now of the University of Michigan, as a class exercise.*

Since the determinant contains ω^2 as a factor every term in its expansion must depend upon at least one of the elements of the main diagonal. Fasten the attention upon any term of the expansion. It can be supposed without loss of generality that it depends upon the first main diagonal element. In the expansion of the determinant this element is multiplied by its minor; consequently we must see if the minor can vanish. The minor is of the same form as the original determinant, and the sum of the elements of its ith row

* An earlier proof was devised by the author, and still another jointly by Professor N. B. MacLean, of the University of Manitoba, and Mr. E. J. Moulton, now of the University of Wisconsin.

is $-\omega^2 - 2m_1/r_{1i}^3$. Consequently every term in the expansion of the minor will contain at least one of the $-\omega^2 - 2m_1/r_{1i}^3$ as a factor. But these elements appear only in the main diagonal of the minor. Hence all terms in the expansion of the minor which do not vanish depend upon at least one element of the main diagonal. In considering our particular term it may be supposed without loss of generality to depend upon the first main diagonal element of the minor. In the expansion of the original determinant the product of these two diagonal elements will be multiplied by the co-factor of the minor of the second order of which they are the main diagonal. This co-factor has properties similar to those of the first minor just considered, and in the same way we prove that at least one of its diagonal elements must be involved in the term in question. That is, the term under consideration depends upon at least three elements of the main diagonal. Continuing in this manner we prove that any term in the final expansion different from zero depends upon all the elements of the main diagonal which are all of the same sign in every one of their terms. Consequently all the terms which do not cancel out in the expansion of the determinant have the sign $(-1)^{\nu+1}$. And we have seen that there is at least one such term, viz., $(-1)^{\nu+1}\omega^{2(\nu+1)}$. Therefore the determinant can not only never vanish, but it can never become less than $\omega^{2(\nu+1)}$ in numerical value.

Since Δ can never vanish for real finite $x_j^{(i)}$ when all the m_j are real and positive, it follows that no real solutions can ever be lost or gained as the m_j vary, and therefore that the number of real solutions of (8) is always exactly $(\nu+1)\,N_\nu = \frac{1}{2}\,(\nu+1)\,!$.

5. Computation of the Solutions of Equations (6). There are well known methods of finding the roots of a single numerical algebraic equation of high degree, but they are not readily applicable to simultaneous equations of high degree. However, when the order of the masses has been chosen, equations (6) will become polynomials in $x_1, \cdots, x_n$ after they have been cleared of fractions. Then by rational processes $n-1$ of the x_i can be eliminated from these equations giving a single equation in the remaining unknown. The solutions of this equation can be found by the usual methods, and the results can be used to eliminate one unknown. By repeated applications of this process to the successively reduced equations, the solutions can all be found. The single one satisfying the conditions of reality of $x_1, \cdots, x_n$ and their order relations is the one desired.

The solutions can also be found by a method closely related to that by

means of which their existence was proved above. Suppose for $m_i = m_i^{(o)}$, $(i = 1, \cdots, n)$, a solution $x_i = x_i^{(o)}$ of equations (6) is known. The $m_i^{(o)}$ are supposed to be zero or positive. Suppose it is desired to find the corresponding solution, that is, the one in which the masses are arranged on the line in the same order, for $m_i = m_i^{(o)} + \mu_i$. Let the corresponding set of the x_i satisfying (6) be $x_i = x_i^{(o)} + \xi_i$, where the ξ_i are functions of $\mu_1, \cdots, \mu_n$ to be determined. Substituting $x_i = x_i^{(o)} + \xi_i$ and $m_i = m_i^{(o)} + \mu_i$ in (6), making use of the notation of (8), expanding as power series in the ξ_i and μ_i (which is always possible since we have shown no $x_i^{(o)}$ can become infinite and no $x_i^{(o)}$ can equal any $x_j^{(o)}$), and remembering that $x_i = x_i^{(o)}$ is a solution of (6) for $m_i = m_i^{(o)}$, we have

$$
(13)\qquad \left\{
\begin{aligned}
&\sum_{j=1}^{n} \frac{\partial \phi_1}{\partial x_j}\xi_j + \sum_{i=2}^{\infty} \frac{1}{i!}\left[\sum_{j=1}^{n} \frac{\partial \phi_1}{\partial x_j}\xi_j\right]^i = -\sum_{j=1}^{n} \frac{\partial \phi_1}{\partial m_j}\mu_j,\\
&\qquad\cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot\\
&\qquad\cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot\\
&\qquad\cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot\\
&\sum_{j=1}^{n} \frac{\partial \phi_n}{\partial x_j}\xi_j + \sum_{i=2}^{\infty} \frac{1}{i!}\left[\sum_{j=1}^{n} \frac{\partial \phi_1}{\partial x_j}\xi_j\right]^i = -\sum_{j=1}^{n} \frac{\partial \phi_n}{\partial m_j}\mu_j,
\end{aligned}
\right.
$$

where $\left[\sum_{j=1}^{n} \frac{\partial \phi_k}{\partial x_j}\xi_j\right]^i$ are the symbolic powers used in connection with the power series expansions of functions of several variables.

The determinant of the terms of the first degree in the ξ_j in equations (13) is the Δ of equation (11) which we have proved can never vanish in this problem. Therefore equations (13) can be uniquely solved for $\xi_1, \cdots, \xi_n$ as power series in $\mu_1, \cdots, \mu_n$ by the method of undetermined coefficients, and these series will converge for $|\mu_i| > 0$ but sufficiently small. Suppose they converge if $|\mu_i| \leqq r$. Keeping the μ_i within this limit a solution $x_i = x_i^{(1)}$ is computed. Then this can be used as a starting point for a second application of the process, which can be repeated as many times as may be desired.

Hence, to find the solution in which the bodies $m_1, \cdots, m_n$ have any finite positive values and lie in a determined order on the line, we may start with m_1, m_2, and m_3 and solve the Lagrangian quintic* which defines their dis-

* Tisserand's *Mécanique Céleste*. vol. 1, p. 155, or Moulton's *Introduction to Celestial Mechanics*, p. 216.

tribution on the line. Then we add an infinitesimal body m_4 and find its position by solving (9) in which now $\nu = 3$. Then we increase the infinitesimal mass, step by step, to the required finite value, and compute the corresponding $x_1, \cdots, x_4$. Making use of the fact that for $|\xi_i| < \rho$ we have for all values of i

$$|\phi_i| < A, \qquad\qquad \Delta \geqq \omega^{2n},$$

where A is a number depending on ρ, ω^2, $m_1, \cdots, m_n$, it can be proved that by this process any finite value m_4 can be reached in a finite number of steps. After the required value of m_4 has been reached the process can be repeated for m_5, etc., to any finite number of bodies. Notwithstanding the fact that this would be very laborious if the number of bodies was large, yet we must regard the problem as completely solved both theoretically and practically.

II. Determination of the Masses when the Positions are Given.

6. Determination of the Masses when n is Even. Suppose ω^2 and the n distinct points on a line, $x_1, \cdots, x_n$, are given, and consider the problem of determining $m_1, \cdots, m_n$ so that the straight line solutions shall exist. There will be no loss of generality in selecting the notation so that $x_1 < x_2 < \cdots < x_n$. With this choice of notation equations (6), which are the necessary and sufficient conditions for the solutions, become

$$(14)\quad \left\{\begin{array}{l}
0 + \dfrac{m_2}{r_{12}^2} + \dfrac{m_3}{r_{13}^2} + \cdots + \dfrac{m_{n-1}}{r_{1\,n-1}^2} + \dfrac{m_n}{r_{1n}^2} = -\omega^2 x_1, \\[2ex]
-\dfrac{m_1}{r_{21}^2} + 0 + \dfrac{m_3}{r_{23}^2} + \cdots + \dfrac{m_{n-1}}{r_{2\,n-1}^2} + \dfrac{m_n}{r_{2n}^2} = -\omega^2 x_2, \\
\quad\cdot \\
\quad\cdot \\
\quad\cdot \\
-\dfrac{m_1}{r_{n-1\,n}^2} - \dfrac{m_2}{r_{n-1\,2}^2} - \dfrac{m_3}{r_{n-1\,3}^2} - \cdots - 0 + \dfrac{m_n}{r_{n-1\,n}^2} = -\omega^2 x_{n-1}, \\[2ex]
-\dfrac{m_1}{r_{n1}^2} - \dfrac{m_2}{r_{n2}^2} - \dfrac{m_3}{r_{n3}^2} - \cdots - \dfrac{m_{n-1}}{r_{n\,n-1}^2} + 0 = -\omega^2 x_n.
\end{array}\right.$$

The m_i enter these equations linearly and are therefore uniquely determined if the determinant

$$(15)\qquad D = \begin{vmatrix} 0 & \frac{1}{r_{12}^2} & \frac{1}{r_{13}^2} & \cdots & \frac{1}{r_{1\,n-1}^2} & \frac{1}{r_{1n}^2} \\ \frac{-1}{r_{21}^2} & 0 & \frac{1}{r_{23}^2} & \cdots & \frac{1}{r_{2\,n-1}^2} & \frac{1}{r_{2n}^2} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \frac{-1}{r_{n-1\,1}^2} & \frac{-1}{r_{n-1\,2}^2} & \frac{-1}{r_{n-1\,3}^2} & \cdots & 0 & \frac{1}{r_{n-1\,n}^2} \\ \frac{-1}{r_{n1}^2} & \frac{-1}{r_{n2}^2} & \frac{-1}{r_{n3}^2} & \cdots & \frac{-1}{r_{n\,n+1}^2} & 0 \end{vmatrix}$$

is distinct from zero. This is a skew-symmetric determinant, and when n is even it is the square of an associated Pfaffian, and, therefore, is not in general zero. Therefore if n is even the masses are in general uniquely determined when ω^2 and $x_1, \cdots, x_n$ are given, though it should be noted that they are not necessarily all positive. When they are negative they have no physical interpretation.

7. Determination of the Masses when n is odd. In this case the skew-symmetric determinant is identically zero, but its first minors of the main diagonal elements, being skew-symmetrical determinants of even order, are in general all distinct from zero; consequently the x_i must satisfy one relation in order that equations (14) shall be consistent. To get this relation add the equation

$$m_1 x_1 + m_2 x_2 + \cdots + m_n x_n = 0,$$

which is a consequence of (14), to the set of equations. In order that they shall be consistent their eliminant

$$
E \equiv \begin{vmatrix}
0 & x_1 & x_2 & \cdots & x_n \\
-x_1 & 0 & \dfrac{1}{r_{21}^2} & \cdots & \dfrac{1}{r_{n1}^2} \\
-x_2 & -\dfrac{1}{r_{12}^2} & 0 & \cdots & \dfrac{1}{r_{n2}^2} \\
\cdot & & & & \cdot \\
\cdot & & & & \cdot \\
\cdot & & & & \cdot \\
-x_n & -\dfrac{1}{r_{1n}^2} & -\dfrac{1}{r_{2n}^2} & \cdots & 0
\end{vmatrix}
$$

must vanish. This is also a skew-symmetrical determinant and is the square of the Pfaffian

$$
(16) \qquad F \equiv \left| \begin{array}{cccc}
x_1 & x_2 & \cdots & x_n \\
 & \dfrac{1}{r_{12}^2} & \cdots & \dfrac{1}{r_{1n}^2} \\
 & & & \cdot \\
 & & & \cdot \\
 & & & \cdot \\
 & & & \dfrac{1}{r_{n\,n-1}^2}
\end{array} \right| = 0.
$$

Equation (16) can be found also by solving any $n-1$ equations of (14) for the corresponding m_i and substituting the solutions in the remaining one. The result is a sum of determinants which can be shown to be the expansion of F multiplied by the square root of the determinant of the coefficients of the $n-1$ m_i in the equations used.

When $F = 0$ is satisfied by $x_1, \cdots, x_n$, equations (14) are consistent. Then, after any m_i has been chosen arbitrarily, the corresponding $n-1$ equations can in general be solved uniquely for the remaining m_j, and the unused equation will be satisfied because $F = 0$.

8. Discussion of Case $n = 3$. When $n = 3$ the determinant D becomes

$$
D \equiv \frac{1}{(r_{12}\, r_{23}\, r_{13})^2} - \frac{1}{(r_{12}\, r_{23}\, r_{13})^2} \equiv 0,
$$

and the Pfaffian F is

$$
(17) \qquad F \equiv \frac{x_1}{r_{23}^2} - \frac{x_2}{r_{13}^2} + \frac{x_3}{r_{12}^2} = 0.
$$

It will now be shown that when any two of x_1, x_2, x_3 are chosen so as to satisfy the conditions $x_1 < x_2 < x_3$, the third is uniquely determined by (17) and these inequalities. From the fact that in this case $r_{13} > r_{12}$, $r_{13} > r_{23}$ it follows that if x_2 is positive then $-x_2/r_{13}^2 + x_3/r_{12}^2$ is positive, and therefore that x_1 must be negative in order that (17) may be satisfied. If x_2 is negative x_1, being less, must also be negative. That is, x_1 is necessarily negative, and similarly x_3 is necessarily positive.

Suppose x_2 and x_3 are chosen and consider F as a function of x_1. Then we have at once

$$(18)\qquad \lim_{x_1 \doteq -\infty} F(x_1) = -\infty\,, \quad \lim_{\epsilon \doteq 0} F(x_2 - \epsilon) = +\infty\,, \quad \frac{\partial F}{\partial x_1} = \frac{1}{r_{23}^2} - \frac{2x_2}{r_{13}^3} + \frac{2x_3}{r_{12}^3}.$$

From the inequalities $x_2 < x_3$ and $r_{12} < r_{13}$ it follows that $\dfrac{\partial F}{\partial x_1}$ is positive for $x_1 < x_2$. Therefore there is one and but one solution of (17) for $x_1 < x_2$ when x_2 and a positive x_3 are chosen. By symmetry there is but one solution of (17) for $x_3 > x_2$ when a negative x_1 and x_2 are chosen.

When x_1 and x_3 are chosen respectively negative and positive but otherwise arbitrarily we have, considering F as a function of x_2,

$$(19)\qquad \lim_{\epsilon \doteq 0} F(x_1 + \epsilon) = +\infty\,, \quad \lim_{\epsilon \doteq 0} F(x_3 - \epsilon) = -\infty\,, \quad \frac{\partial F}{\partial x_2} = \frac{2x_1}{r_{23}^3} - \frac{1}{r_{13}^2} - \frac{2x_3}{r_{12}^3} < 0.$$

Therefore there is one and but one solution of (17) for x_2 satisfying the inequalities $x_1 < x_2 < x_3$.

Suppose a negative x_1, a positive x_3, and m_2 are given arbitrarily and that x_2 is defined by (17). From (14) we have

$$(20)\qquad \begin{cases} m_1 = -\,r_{13}^2\left[\dfrac{m_2}{r_{23}^2} - \omega^2 x_3\right], \\[2ex] m_3 = -\,r_{13}^2\left[\dfrac{m_2}{r_{12}^2} + \omega^2 x_1\right]. \end{cases}$$

Since x_1 is negative and x_3 positive it follows from (20) that m_2 can be taken so small that m_1 and m_3 are positive. If $-x_1$ is not equal to x_3, then a positive m_2 can be determined so that m_1 and m_3 shall both be positive, one positive and one negative, or both negative; but m_1, m_2, and m_3 can not all be negative.

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ON SEMI-ANALYTIC FUNCTIONS OF TWO VARIABLES

By Maxime Bôcher

When functions of two variables are considered, it is customary either to assume that they are analytic in their two independent arguments, or, both arguments being restricted to real values, merely to impose certain conditions of continuity and differentiability upon them. An intermediate case is, however, not without its importance,* namely that in which, while the function is continuous in the two independent variables, it is analytic in only one of them. My object in the present note is to state and prove some of the fundamental theorems relating to this case. The extensions to the case of functions of more than two variables would seem to be obvious, and to present no serious difficulty.

The real variable x we restrict to a connected interval X of the x-axis. This interval may be finite or infinite, open or closed. The complex variable λ shall be restricted to a two-dimensional continuum Λ of the λ-plane, that is to a connected region every point of which is an interior point. For convenience we assume that Λ is simply connected.† The totality of points (x, λ) which are such that x lies in X and λ in Λ we speak of as the region (X, Λ).

Lemma 1. *If $f(x, \lambda)$ is continuous in (x, λ) and analytic in λ throughout the region (X, Λ), then, as $\Delta\lambda$ approaches zero, the difference-quotient*

$$D = \frac{f(x, \lambda + \Delta\lambda) - f(x, \lambda)}{\Delta\lambda}$$

converges uniformly to the value $\partial f/\partial\lambda$ throughout any closed sub-region†† *(X', Λ') of (X, Λ).*

Let us consider a regular closed curve C every point of which lies in Λ while no point of it lies in Λ' and which surrounds Λ' once in the positive

* For instance in the theory of linear differential or integral equations. Cf. for example, Kneser's paper in volume 58 of the *Mathematische Annalen*, in particular pages 109-116.

† This restriction is essential in Theorem II. The reader may readily satisfy himself that all the other theorems are equally true for multiply connected regions.

†† The term sub-region, as used in this paper means a region consisting of some *or all* points of the given region.

direction. Then by Cauchy's integral formula we have for all points of (X', Λ')

$$f(x, \lambda) = \frac{1}{2\pi i}\int \frac{f(x, t)}{t - \lambda}\, dt,$$

$$\frac{\partial f(x, \lambda)}{\partial \lambda} = \frac{1}{2\pi i}\int \frac{f(x, t)}{(t - \lambda)^2}\, dt,$$

the integral in each case being extended in the positive direction around C. Using the first of these formulæ, we find that, when $|\Delta\lambda|$ is less than the smallest distance k between the boundary of Λ' and the curve C, for all points of (X', Λ') the formula holds:

$$D = \frac{1}{2\pi i}\int \frac{f(x, t)}{(t - \lambda - \Delta\lambda)(t - \lambda)}\, dt.$$

We may therefore write

$$D - \frac{\partial f}{\partial \lambda} = \frac{\Delta\lambda}{2\pi i}\int \frac{f(x, t)}{(t - \lambda - \Delta\lambda)(t - \lambda)^2}\, dt.$$

The values of t and λ which occur in this formula satisfy the inequality

$$|t - \lambda| \geqq k.$$

Let us now restrict $\Delta\lambda$ so that

$$|\Delta\lambda| < \frac{k}{2}.$$

Then

$$|t - \lambda - \Delta\lambda| \geqq |t - \lambda| - |\Delta\lambda| \geqq \frac{k}{2}.$$

If we denote by M the upper limit of $|f(x, \lambda)|$ in the closed region (X', C) and by l the length of the curve C, we have

$$\left| D - \frac{\partial f}{\partial \lambda} \right| \leqq \frac{Ml}{\pi k^3}\, |\Delta\lambda|,$$

from which the uniform convergence of D towards $\frac{\partial f}{\partial \lambda}$ follows.

Lemma 2. *If $f(x, \lambda)$ is continuous in (x, λ) throughout the region (X, Λ) and has a derivative $\partial f/\partial x$ which is also continuous throughout this region, then, as Δx approaches zero, the difference quotient*

$$\frac{f(x+\Delta x, \lambda) - f(x, \lambda)}{\Delta x}$$

converges uniformly to the value $\partial f/\partial x$, *throughout any closed sub-region* (X', Λ') *of* (X, Λ).

If we split f into its real and pure imaginary parts:

$$f(x, \lambda) = \phi(x, \lambda) + i\psi(x, \lambda),$$

we need prove our theorem only for the two real functions ϕ and ψ, so that, since these functions satisfy precisely the same conditions as f, it will clearly be sufficient to prove our theorem on the supposition that f is real. In this case our difference quotient may be written

$$\frac{\partial f(x+\theta\Delta x, \lambda)}{\partial x} \qquad (0 < \theta < 1).$$

No matter what constant ϵ we choose, we can, on account of the continuity of $\partial f/\partial x$, find a constant δ so small that, when $|\Delta x| < \delta$,

$$\left| \frac{\partial f(x+\Delta x, \lambda)}{\partial x} - \frac{\partial f(x, \lambda)}{\partial x} \right| < \epsilon$$

throughout the region (X', Λ').* Consequently when $|\Delta x| < \delta$ our difference quotient differs from the derivative by a quantity which in absolute value is less than ϵ; and thus our lemma is proved.

THEOREM I. *If* $f(x, \lambda)$ *is continuous in* (x, λ) *and analytic in* λ *throughout the region* (X, Λ), *the same will be true of* $\partial f/\partial \lambda$.

We know from the elements of the theory of functions of a complex variable that $\partial f/\partial\lambda$ is analytic in λ. It remains then merely to show that at an arbitrary point (x', λ') of (X, Λ) $\partial f/\partial\lambda$ is continuous in (x, λ). Let us choose a closed sub-region (X', Λ') so as to include the point (x', λ') and so that x' and λ' are interior points of X' and Λ' respectively, except that when x' is an end-point of X it shall also be an end-point of X'. If we restrict $\Delta\lambda$ to be in absolute value less than the least distance between the boundaries of Λ and Λ', it is clear that the difference-quotient D of Lemma I is a continuous function of (x, λ) throughout the region (X', Λ'). Since, when $\Delta\lambda$ approaches

* We are here making use of the fact that a function of two variables continuous throughout a closed region is uniformly continuous there. This theorem, and the proof ordinarily given of it, hold in the case we are here considering where one variable is real the other complex.

zero, D approaches its limit uniformly throughout (X', Λ'), it follows that throughout this region, and consequently in (x', λ'), the limit $\partial f/\partial\lambda$ is also continuous, as was to be proved.

COROLLARY. *The functions*

$$\frac{\partial^n f}{\partial\lambda^n} \qquad (n = 2, 3, \cdots)$$

are continuous in (x, λ) *and analytic in* λ *throughout the region* (X, Λ).

THEOREM II. *If* $f(x, \lambda)$ *is continuous in* (x, λ) *and analytic in* λ *throughout the region* (X, Λ), *the same will be true of the function*

$$\Phi(x, \lambda) = \int_{\lambda_0}^{\lambda} f(x, \mu)\, d\mu$$

λ_0 *being an arbitrary point in* Λ.

That this function is analytic in λ is known from the elements of the theory of functions of a complex variable. To prove it continuous in (x, λ) at an arbitrary point (x', λ') of (X, Λ) we form the difference

$$\Phi(x' + \Delta x, \lambda' + \Delta\lambda) - \Phi(x', \lambda') = [\Phi(x' + \Delta x, \lambda' + \Delta\lambda) - \Phi(x', \lambda' + \Delta\lambda)] + [\Phi(x', \lambda' + \Delta\lambda) - \Phi(x', \lambda')].$$

Since Φ is an analytic, and therefore continuous, function of λ at the point (x', λ'), no matter how small the positive constant ϵ may be, a constant δ_1 can be so chosen that when $|\Delta\lambda| < \delta_1$

$$|\Phi(x', \lambda' + \Delta\lambda) - \Phi(x', \lambda')| < \frac{\epsilon}{2}.$$

Let us denote by l the length of a regular curve C lying in Λ and connecting λ_0 with λ'. About λ' as centre construct a circle which lies wholly in Λ and whose radius δ_2 is less than l. Then when $|\Delta\lambda| < \delta_2$, we may use as the path of integration from λ_0 to $\lambda' + \Delta\lambda$ the curve C and a straight line from λ' to $\lambda' + \Delta\lambda$. The length of this path is less than $2l$.

We have

$$\Phi(x' + \Delta x, \lambda' + \Delta\lambda) - \Phi(x', \lambda' + \Delta\lambda) = \int_{\lambda_0}^{\lambda + \Delta\lambda} [f(x' + \Delta x, \mu) - f(x', \mu)]\, d\mu.$$

Consider now the closed region T which consists of the circle of radius δ_2 about λ' and so much of the curve C as does not already lie in this circle.

Consider also any closed sub-interval X' of X of which x' is an interior point, or, when x' is an end-point of X, of which x' is an end-point. Since f is continuous, and therefore uniformly continuous, in (X', T), a constant δ_3 can be found such that when $|\Delta x| < \delta_3$ and $x' + \Delta x$ lies in X

$$|f(x' + \Delta x, \mu) - f(x', \mu)| < \frac{\epsilon}{4l}$$

throughout the region (X', T). Consequently, since our path of integration lies in T and is of length less than $2l$, it follows that when $|\Delta\lambda| < \delta_2$, $|\Delta x| < \delta_3$ and $x' + \Delta x$ lies in X

$$|\Phi(x' + \Delta x, \lambda' + \Delta\lambda) - \Phi(x', \lambda' + \Delta\lambda)] < \frac{\epsilon}{2}.$$

Combining this with the similar inequality found above, and denoting by δ the smallest of the three quantities δ_1, δ_2, δ_3, we see that when $|\Delta x| < \delta$, $|\Delta\lambda| < \delta$ and $x' + \Delta x$ lies in X

$$|\Phi(x' + \Delta x, \lambda' + \Delta\lambda) - \Phi(x', \lambda')| < \epsilon,$$

and the continuity of Φ is established.

THEOREM III. *If $f(x,\lambda)$ is continuous in (x,λ) and analytic in λ throughout the region (X, Λ) and has a derivative $\partial f/\partial x$ which is continuous in (x, λ) throughout this region, then this derivative is also analytic in λ throughout this region.*

Consider any regular closed curve C in Λ. If x and $x + \Delta x$ both lie in X, we have by Cauchy's integral theorem

$$\int \frac{f(x + \Delta x, \lambda) - f(x, \lambda)}{\Delta x}\, d\lambda = 0,$$

the integral being extended around C. A reference to Lemma 2 shows that, as Δx approaches zero, the integrand here converges uniformly to $\partial f/\partial x$ for all values of λ on C. Consequently

$$\int \frac{\partial f}{\partial x}\, d\lambda = 0,$$

the path of integration again being C. But since C is any closed regular curve in Λ, it follows from Morera's Theorem * that $\partial f/\partial x$ is analytic in λ.

* Cf. Osgood, *Funktionentheorie*, Bd. I, p. 256.

THEOREM IV. *If $f(x, \lambda)$ is continuous in (x, λ) and analytic in λ throughout the region (X, Λ), the same will be true of the function*

$$F(x, \lambda) = \int_c^x f(a, \lambda)\, da,$$

c being any fixed point in X.

We prove first that F is analytic in λ. For this purpose it is sufficient to prove that it has a derivative with regard to λ at every point of (X, Λ). We form the difference quotient

$$\frac{F(x, \lambda + \Delta\lambda) - F(x, \lambda)}{\Delta\lambda} = \int_c^x \frac{f(a, \lambda + \Delta\lambda) - f(a, \lambda)}{\Delta\lambda}\, da.$$

The integrand here is the continuous function D of Theorem I, which, as we there saw, approaches $\partial f/\partial\lambda$ uniformly as $\Delta\lambda$ approaches zero, provided we restrict ourselves to a closed sub-region of (X, Λ), as we clearly have a right to do here. From a fundamental theorem on uniform convergence it follows that the right hand side of the last written formula approaches a limit, namely

$$\int_c^x \frac{\partial f(a, \lambda)}{\partial\lambda}\, da.$$

Thus the analytic character of F in λ is established.

Consider now the difference

$$\begin{aligned} F(x' + \Delta x, \lambda' + \Delta\lambda) - F(x', \lambda') = &[F(x' + \Delta x, \lambda' + \Delta\lambda) - F(x', \lambda' + \Delta\lambda)] \\ &+ [F(x', \lambda' + \Delta\lambda) - F(x', \lambda')] \end{aligned}$$

where (x', λ') is an arbitrary point of (X, Λ) and Δx, $\Delta\lambda$ are so restricted that $(x' + \Delta x, \lambda' + \Delta\lambda)$ also lies in (X, Λ).

Since $F(x, \lambda)$ is analytic, and therefore continuous, in λ at the point (x', λ'), it follows that, however small ϵ may be, a positive constant δ_1 can be so chosen that when $|\Delta\lambda| < \delta_1$ the second parenthesis on the right of the last written formula is in absolute value less than $\frac{1}{2}\epsilon$. The first parenthesis may be written.

$$\int_{x'}^{x' + \Delta x} f(a, \lambda' + \Delta\lambda)\, da$$

and it is readily seen that a constant δ_2 can be so chosen that when $|\Delta x| < \delta_2$ this integral is in absolute value less than $\frac{1}{2}\epsilon$. If we denote the smaller of the two constants δ_1, δ_2 by δ, it follows that when $|\Delta x| < \delta$, $|\Delta\lambda| < \delta$

$$|F(x' + \Delta x, \lambda' + \Delta\lambda) - F(x', \lambda')| < \epsilon,$$

and thus the continuity of F at the point (x', λ') is established. The proof just given establishes also the

COROLLARY. *The derivative with regard to λ of F may be found by differentiating under the integral sign.*

THEOREM V. *If we develop by Taylor's Theorem about a point λ_0 of Λ a function $f(x, \lambda)$ which is continuous in (x, λ) and analytic in λ throughout (X, Λ), and if we denote by Λ' a circle, inclusive of its circumference, which lies wholly in Λ and whose centre is at λ_0, then the coefficients of this development are functions of x which are continuous in X, and the development converges uniformly in (x, λ) throughout the region (X', Λ') where X' is any closed sub-region of X.*

Let the development here referred to be

$$(1) \qquad f(x, \lambda) = f_0(x) + f_1(x)(\lambda - \lambda_0) + f_2(x)(\lambda - \lambda_0)^2 + \cdots.$$

That the coefficients are continuous functions of x is an immediate consequence of the corollary to Theorem I, since $f_n(x)$ differs only by a constant factor from the value of $\partial^n f/\partial\lambda^n$ at the point λ_0.

In order to establish the uniform convergence of (1) we proceed as follows.* Let ρ be the radius of Λ'. With λ_0 as centre construct a second circle whose circumference lies wholly in Λ but whose radius r is greater than ρ. The points on the circumference of this circle form a closed region which we will call T. Let M be the maximum of $|f(x, \lambda)|$ in the closed region (X', T). Then by a well-known theorem on power-series (Cf. Forsyth, Theory of Functions, 2nd Ed. p. 34)

$$M \geqq |f_n(x)| r^n.$$

Consequently throughout (X', Λ') the terms of (1) do not exceed in absolute value the corresponding terms of the series

$$M + M\frac{\rho}{r} + M\left(\frac{\rho}{r}\right)^2 + \cdots.$$

Since this is a convergent series of positive constant terms, the uniform convergence of (1) in (X', Λ') is established.

* For a more general formulation of this part of the theorem Cf. the Lemma on page 40 of my *Introduction to the Study of Integral Equations. Cambridge Tracts in Mathematics and Mathematical Physics*, No. 10.

We have at the same time established the

COROLLARY. *Not only* (1) *but also the series*

$$|f_0(x)| + |f_1(x)|\,|\lambda - \lambda_0| + |f_2(x)|\,|\lambda - \lambda_0|^2 + \cdots$$

is uniformly convergent in (x, λ) *throughout the region* (X', Λ').

THEOREM VI. *If* $f(x, \lambda)$ *is continuous in* (x, λ) *and analytic in* λ *throughout the region* (X, Λ) *and possesses a derivative* $\partial f/\partial x$ *continuous in* (x, λ) *throughout this region, and if we develop* $f(x, \lambda)$ *by Taylor's Theorem about a point* λ_0 *of* Λ, *the coefficients of this series have derivatives continuous throughout* X, *and, if we differentiate the series term by term with regard to* x, *we obtain the Taylor's development of* $\partial f/\partial x$ *about the point* λ_0.*

If we write the Taylor's Series, as above, in the form (1), we have by a well-known formula of Cauchy

$$f_n(x) = \frac{1}{2\pi i}\int \frac{f(x, t)}{(t - \lambda_0)^{n+1}}\,dt$$

where the integral may be extended in the positive direction about a circumference C lying in Λ and having λ_0 as centre. We form the difference-quotient

$$\frac{f_n(x + \Delta x) - f_n(x)}{\Delta x} = \frac{1}{2\pi i}\int \frac{f(x + \Delta x, t) - f(x, t)}{\Delta x}\,\frac{dt}{(t - \lambda_0)^{n+1}}.$$

A reference to Lemma 2 shows that, as Δx approaches zero, the integrand converges uniformly in t for all points on C to the limit

$$\frac{\partial f(x, t)}{\partial x}\cdot\frac{1}{(t - \lambda_0)^{n+1}}.$$

Consequently $f_n(x)$ has a derivative given by the formula

$$f_n'(x) = \frac{1}{2\pi i}\int \frac{\partial f(x, t)}{\partial x}\cdot\frac{dt}{(t - \lambda_0)^{n+1}}.$$

This is, however, precisely the formula for the coefficient of $(\lambda - \lambda_0)^n$ in the Taylor's development of $\partial f(x, \lambda)/\partial x$ about the point λ_0. Consequently that part of our theorem which says that this last development may be obtained by differentiating the series (1) term by term with regard to x is established. The fect that $f_n'(x)$ is continuous throughout X now follows at once by a ref-

* That $\partial f(x,\lambda)/\partial x$ is analytic in λ and therefore admits of a Taylor's development, we see from Theorem III.

erence to Theorem V, since that theorem may be applied to the development of $\partial f(x, \lambda)/\partial x$.

THEOREM VII. *If we denote by Λ a circle, exclusive of its circumference, whose centre is λ_0 and whose radius is R, and if the functions $f_0(x), f_1(x), \cdots$ are continuous in X and a constant M exists such that throughout X*

$$|f_n(x)|\, R^n < M \qquad (n = 0, 1, 2, \cdots),$$

then throughout the region (X, Λ) the series

$$(1) \qquad f_0(x) + f_1(x)(\lambda - \lambda_0) + f_2(x)(\lambda - \lambda_0)^2 + \cdots$$

represents a function continuous in (x, λ) and analytic in λ.

For let Λ' be a circle, *inclusive* of its circumference, with centre at λ_0 and radius $r < R$. For all points of Λ' we have

$$|f_n(x)|\,|\lambda - \lambda_0|^n \leqq |f_n(x)|\, r^n < M\left(\frac{r}{R}\right)^n.$$

Consequently, for all such points, the terms of (1) are in absolute value less than the corresponding terms of the series

$$M + M\frac{r}{R} + M\left(\frac{r}{R}\right)^2 + \cdots$$

and, since this is a convergent series of positive constant terms, it follows that (1) is uniformly convergent in (x, λ) throughout the region (X, Λ'). Accordingly (1) represents a function of (x, λ) continuous throughout this region, and consequently, since Λ' can be made to include any arbitrarily chosen point of Λ, the function represented by (1) is continuous in (x, λ) throughout (X, Λ). That it is analytic in λ throughout this region is obvious from the fact that (1) is a power-series in $\lambda - \lambda_0$.

The above theorem remains true if by Λ is understood the same circle as above *including* its circumference, and the condition referring to the inequality in Theorem VII is replaced by the condition that the series

$$|f_0(x)| + |f_1(x)|\, R + |f_2(x)|\, R^2 + \cdots$$

be uniformly convergent in X. Here series (1) can be seen to be uniformly convergent in (X, Λ). This modification of Theorem VII is less general than that theorem except in the one particular that the circumference of Λ is included.

MUNICH, GERMANY
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SOME THEOREMS CONCERNING SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EXPRESSIONS

By W. J. Berry

The investigations which led to the preparation of this paper, undertaken somewhat over a year ago at the suggestion of Professor Maxime Bôcher, were carried on, in part, under his direction, and the writer wishes to express his appreciation of the many illuminating criticisms received from him during the course of the work. Thanks are also due to Dr. Frank Irwin for permission to make use of his paper on Invariants of Linear Differential Expressions*, and for his kindness in reading and commenting upon the manuscript. Professor Elijah Swift has put the writer under obligations by helpful suggestions as to the form of presentation. The results obtained are in the nature of generalizations of theorems derived by Dr. Irwin in the paper referred to, and though the discussion is here limited to systems of linear partial differential expressions of the *second* order, the extension to systems of expressions of any order is immediate. The properties considered are five in number:—the existence of a system of multipliers; the existence of an adjoint, including the condition that a given system be self-adjoint; the existence of a Lagrange Identity; the existence of a Green's Theorem; and the existence of invariants.

The general system, L, of n second order linear partial differential expressions in n dependent and p independent variables may be written

$$L_i(u_1, \cdot \cdot \cdot, u_n) \equiv$$

$$\sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{r=1}^{p} l_{k,r}^{(i,j)} \frac{\partial^2 u_j}{\partial x_k \, \partial x_r} + \sum_{j=1}^{n} \sum_{k=1}^{p} l_k^{(i,j)} \frac{\partial u_j}{\partial x_k} + \sum_{j=1}^{n} l^{(i,j)} u_j, \quad i = 1, \cdot \cdot \cdot, n.$$

That the functions occurring in the course of the discussion may have all the necessary continuity, we shall demand:

1–) that in some p– dimensional region Ω of the space R_p defined by the independent variables, all the coefficients with double subscripts shall be continuous, with continuous partial derivatives of the first two orders with

* *Proc. Am. Acad. of Arts and Sciences*, vol. 44, no. 1, Nov. 1908.

respect to each of their arguments, and with continuous second order cross derivatives of all possible types;

2–) that all the coefficients with a single subscript shall be continuous in Ω, and shall there have continuous partial derivatives of the first order with respect to each of their arguments;

3–) that all coefficients without subscripts shall be continuous in Ω.

We shall further agree that hereafter throughout this paper, $l^{(i,j)}_{k,r}$ shall equal $l^{(i,j)}_{r,k}$ identically.

Definition. A set, v, of functions $v_i(x_1, \cdots, x_p)$, $i = 1, \cdots, n$, is said to form a system of multipliers for the system L, if

$$\sum_{i=1}^{n} v_i L_i \equiv \sum_{k=1}^{p} \frac{\partial S_k}{\partial x_k},$$

where

$$S_k \equiv \sum_{j=1}^{n} \sum_{r=1}^{p} s^{(k,j)}_r \frac{\partial u_j}{\partial x_k} + \sum_{j=1}^{n} s u^{(k,j)}_j$$

Concerning these coefficients we shall demand that they be functions of $x_1, \cdots, x_p$ continuous in Ω, and having there continuous partial derivatives of the first order with respect to each of their arguments.

The system L may also be written

$$L_i (u_1, \cdots, u_n) \equiv \sum_{j=1}^{n} L_{i,j} (u_j), \qquad i = 1, \cdots, n,$$

where

$$L_{i,j} \equiv \sum_{k=1}^{p} \sum_{r=1}^{p} l^{(i,j)}_{k,r} \frac{\partial^2 u_j}{\partial x_k \partial x_r} + \sum_{k=1}^{p} l^{(i,j)}_k \frac{\partial u_j}{\partial x_k} + l^{(i,j)} u_j.$$

The $L_{i,j}$ are, then, linear partial differential expressions of the second order in *one* dependent and p independent variables,—that is to say, expressions of precisely the sort treated by Dr. Irwin in his paper. If the system v is to form a set of multipliers for the system L, it is necessary that

$$\sum_{i=1}^{n} v_i L_i \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} v_i L_{i,j} \equiv \sum_{k=1}^{p} \frac{\partial S_k}{\partial x_k}.$$

Since this is an identity, the terms in u_j on the left must be identically equal to those on the right, hence

$$\sum_{i=1}^{n} v_i\, L_{i,j}(u_j) \equiv \sum_{k=1}^{p} \frac{\partial S_{k,j}}{\partial x_k},$$

where

$$S_{k,j} \equiv \sum_{r=1}^{p} s_r^{(k,j)} \frac{\partial u_j}{\partial x_r} + s u_j^{(k,j)}.$$

Equating coefficients from the two sides of this identity leads to the following equivalent set of necessary conditions for the existence of the system v:

$$2 \sum_{i=1}^{n} v_i\, l_{k,r}^{(i,j)} \equiv s_r^{(r,j)} + s_k^{(r,j)},$$

$$\sum_{i=1}^{n} v_i\, l_k^{(i,j)} \equiv \sum_{r=1}^{p} \frac{\partial s_k^{(r,j)}}{\partial x_r} + s^{(k,j)},$$

$$\sum_{i=1}^{n} v_i\, l^{(i,j)} \equiv \sum_{k=1}^{p} \frac{\partial s^{(k,j)}}{\partial x_k}, \qquad j = 1, \cdots n.$$

Operating on each identity of the first type with $\frac{\partial^2}{\partial x_k \partial x_r}$; on each identity of the second type with $-\frac{\partial}{\partial x_k}$; on each identity of the third type with 1, and adding all the resulting expressions for which j has a common value, gives the system of identities

$$\sum_{i=1}^{n} \sum_{k=1}^{p} \sum_{r=1}^{p} \frac{\partial^2}{\partial x_k \partial x_r} \left[l_{k,r}^{(i,j)} v_i \right] - \sum_{i=1}^{n} \sum_{k=1}^{p} \frac{\partial}{\partial x_k} \left[l_k^{(i,j)}\, v_i \right] + \sum_{i=1}^{n} l^{(i,j)} v_i$$

$$\equiv \sum_{k=1}^{p} \sum_{r=1}^{p} \frac{\partial^2 s_r^{(k,j)}}{\partial x_k\, \partial x_r} - \sum_{k=1}^{p} \left\{ \sum_{r=1}^{p} \frac{\partial^2 s_r^{(k,j)}}{\partial x_k\, \partial x_r} + \frac{\partial s^{(k,j)}}{\partial x_k} \right\} + \sum_{k=1}^{p} \frac{\partial s^{(k,j)}}{\partial x_k} \equiv 0,$$
$$j = 1, \cdots, n.$$

The expressions at the left form a system, M, of second order linear partial differential expressions

$$M_i(v_1, \cdots, v_n)$$

$$\equiv \sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{r=1}^{p} \frac{\partial^2}{\partial x_k \partial x_r} \left[l_{k,r}^{(j,i)} v_j \right] - \sum_{j=1}^{n} \sum_{k=1}^{p} \frac{\partial}{\partial x_k} \left[l_k^{(j,i)} v_j \right] + \sum_{j=1}^{n} l^{(j,i)} v_j$$

$$\equiv \sum_{j=1}^{n} M_{i,j}(v_j),$$

$$i = 1, \cdots, n,$$

where $M_{i,j}(v_j)$ is the adjoint of $L_{j,i}(u_i)$ under the definition laid down by Dr. Irwin.*

Definition. The system of second order linear partial differential expressions M is said to be the *adjoint system* of the system L.

It is necessary, then, in order that the system v form a system of multipliers for the system L, that it be a set of solutions of the system of differential equations

$$M_i(v_1, \cdots, v_n) = 0, \qquad i = 1, \cdots, n.$$

This necessary condition is also sufficient, for if we have given the v_i as solutions of this system of differential equations, and choose certain of the coefficients $s_r^{(k,j)}$ at will, say those for which $k > r$†, then it will be possible to determine all the other coefficients so as to satisfy all the necessary conditions of the first two types.

It follows that

$$\sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{r=1}^{p} \frac{\partial^2}{\partial x_k \, \partial x_r} \left[l_{k,r}^{(j,i)} v_j \right] - \sum_{j=1}^{n} \sum_{k=1}^{p} \frac{\partial}{\partial x_k} \left[l^{(j,i)} v_j \right] \equiv - \sum_{k=1}^{p} \frac{\partial s^{(k,i)}}{\partial x_k}, \qquad i = 1, \cdots, n,$$

and the necessary conditions of the third type are automatically satisfied. We are thus led to

Theorem 1. A necessary and sufficient condition that the system of functions v form a set of multipliers for the system of second order linear partial differential expressions L, is that it be a set of solutions of the system of second order linear partial differential *equations*

$$M_i = 0, \qquad i = 1, \cdots, n,$$

in which the left-hand members are the expressions of the system M, adjoint to L.

The idea of "multiplier", as here defined, represents the extension to systems of second order linear *partial* differential *expressions* of analogous concepts introduced by Jacobi in connection with systems of first order linear homogeneous *ordinary* differential *equations*‡.

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* Loc. cit., page 8.

† This is, of course, not the only choice possible, but it agrees with that made by Dr. Irwin in the simpler case; loc. cit., page 9.

‡ *Encyklopädie der Math. Wiss.*, Bd. II, A 4 b, § 12.

The adjoint system, consisting, like the original system, of second order linear partial differential expressions, may be written with its own coefficients,

$$M_i(v_1, \cdots, v_n) \equiv \sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{r=1}^{p} m_{k,r}^{(i,j)} \frac{\partial^2 v_j}{\partial x_k \, \partial x_r} + \sum_{j=1}^{n} \sum_{k=1}^{p} m_k^{(i,j)} \frac{\partial v_j}{\partial x_k} + \sum_{j=1}^{n} m^{(i,j)} v_j,$$

$$i = 1, \cdots, n,$$

these coefficients being connected with those of L by the relations

$$m_{k,r}^{(i,j)} \equiv l_{k,r}^{(j,i)},$$

$$m_k^{(i,j)} \equiv 2 \sum_{r=1}^{p} \frac{\partial l_{k,r}^{(j,i)}}{\partial x_r} - l_k^{(j,i)},$$

$$m^{(i,j)} \equiv \sum_{k=1}^{p} \sum_{r=1}^{p} \frac{\partial^2 l_{k,r}^{(j,i)}}{\partial x_k \partial x_r} - \sum_{k=1}^{p} \frac{\partial l_k^{(j,i)}}{\partial x_k} + l^{(j,i)},$$

or in more symmetrical form,

$$m_{k,r}^{(i,j)} \equiv l_{k,r}^{(j,i)},$$

$$\sum_{r=1}^{p} \frac{\partial m_{k,r}^{(i,j)}}{\partial x_r} - m_k^{(i,j)} \equiv - \sum_{r=1}^{p} \frac{\partial l_{k,r}^{(j,i)}}{\partial x_r} + l_k^{(j,i)},$$

$$\sum_{k=1}^{p} \frac{\partial m_k^{(i,j)}}{\partial x_k} - 2m^{(i,j)} \equiv \sum_{k=1}^{p} \frac{\partial l_k^{(j,i)}}{\partial x_k} - 2l^{(j,i)}.$$

This same symmetry shows that the adjoint relation is mutual,—that is to say, if the system M is adjoint to L, L is also adjoint to M.

Definition. By a self-adjoint system we shall agree to mean one such that when its adjoint has been formed the coefficients will be found to be identical with those of the original system.

The truth of the following theorem is at once evident.

Theorem 2. Necessary and sufficient conditions that a system L of second order linear partial differential expressions be self-adjoint are:

1) $l^{(i,j)} \equiv l^{(j,i)}$, whatever the lower indices;

2) $l_k^{(i,j)} \equiv \sum_{r=1}^{p} \frac{\partial l_{k,r}^{(i,j)}}{\partial x_r}, \qquad i, j = 1, \cdots, n; \; k = 1, \cdots, p.$

For the special case in which the coefficients of L are constants, this second condition becomes $l_k^{(i,j)} \equiv 0$,

$$\sum_{i=1}^{n}\left\{v_i L_i(u_1, \cdots, u_n) - u_i M_i(v_1, \cdots, v_n)\right\} \equiv$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{ v_{j_i} L_{ji}(u_i) - u_i M_{i,j}(v_j)\right\}.$$

Since $L_{j,i}(u_i)$ and $M_{i,j}(v_j)$ are adjoint expressions they satisfy the relation

$$v_j L_{j,i}(u_i) - u_i M_{i,j}(v_j) \equiv \sum_{k=1}^{p} \frac{\partial R_k^{(i,j)}}{\partial x_k}, \qquad i, j = 1, \cdots, n,$$

where the $R_k^{(i,j)}$ are bilinear in u_i and v_j and their partial derivatives of the first order.* Hence

$$\sum_{i=1}^{n}\left\{v_i L_i - u_i M_i\right\} \equiv \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{p} \frac{\partial R_k^{(i,j)}}{\partial x_k} \equiv \sum_{k=1}^{p}\frac{\partial R_k}{\partial x_k},$$

where the R_k are bilinear in all the u_i and v_i and their partial derivatives of the first order.

Definition. A relation of the form

$$\sum_{i=1}^{n}\left\{v_i L_i - u_i M_i\right\} \equiv \sum_{k=1}^{p}\frac{\partial R_k}{\partial x_k},$$

in which L and M are adjoint systems and the functions R_k are bilinear in the u_i and v_i and their partial derivatives of the first order, is said to be *Lagrange's Identity* for the systems u and v.

It will be noted that Lagrange's Identity furnishes a new and simple proof for the second part of Theorem 1, for if the system v is a set of solutions of $M_i(v_1, \cdots, v_n) = 0, \quad i = 1, \cdots, n,$ then

$$M_i(v_1, \cdots, v_n) \equiv 0, \quad \text{and} \quad \sum_{i=1}^{n} v_i L_i \equiv \sum_{k=1}^{p}\frac{\partial R_k}{\partial x_k},$$

where the R_k satisfy all the conditions imposed on the S_k of the earlierr poofs. Therefore v is a system of multipliers for L, and, from the symmetry of Lagrange's Identity, u is a system of multipliers for M.

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* See Irwin loc. cit., page 10.

Theorem 3. If between two systems, L and N, each consisting of n second order linear partial differential expressions in n dependent and p independent variables, there exists a relation of the form of Lagrange's Identity, L and N are adjoint systems.

For, let M be the system adjoint to L, then

$$\sum_{i=1}^{n} \left\{ v_i L_i(u_1, \cdots, u_n) - u_i M_i(v_1, \cdots, v_n) \right\} \equiv \sum_{k=1}^{p} \frac{\partial R_k}{\partial x_k}.$$

By hypothesis

$$\sum_{i=1}^{n} \left\{ v_i L_i(u_1, \cdots, u_n) - u_i N_i(v_1, \cdots, v_n) \right\} \equiv \sum_{k=1}^{p} \frac{\partial T_k}{\partial x_k},$$

$$\sum_{i=1}^{n} u_i(N_i - M_i) \equiv \sum_{k=1}^{p} \left(\frac{\partial R_k}{\partial x_k} - \frac{\partial T_k}{\partial x_k}\right) \equiv \sum_{k=1}^{p} \frac{\partial}{\partial x_k}\,(R_k - T_k).$$

The system u is accordingly a system of multipliers for $(N - M)$. If $\mathfrak{M}$ be the adjoint system to $(N - M)$, u is a set of solutions for $\mathfrak{M}$. But the u_i are entirely arbitrary; hence $\mathfrak{M}_i \equiv 0$, and consequently

$$N_i - M_i \equiv 0, \qquad i = 1, \cdots, n,$$

which proves the theorem.

Definition. The expression obtained by integrating Lagrange's Identity,

$$\iint_{\Omega}\cdots\int \left\{ \sum_{i=1}^{n} (v_i L_i - u_i M_i) \right\} dx_1 \cdots dx_p \equiv \int_S \sum_{k=1}^{p} R_k \cos a_k \, dS,$$

is defined as a Green's Theorem for the system L, S being the $(p - 1)$-way spread which is the boundary of the region Ω.

If the system L is self-adjoint, the expression $L_{i,j}\,(u_j)$ is self-adjoint. Such an expression satisfies a three term form of Lagrange's Identity:*

$$v_j L_{j,i}(u_i) - \sum_{k=1}^{p} \frac{\partial P_k^{(i,j)}}{\partial x_k} \equiv u_i M_{i,j}(v_j) - \sum_{k=1}^{p} \frac{\partial Q_k^{(i,j)}}{\partial x_k}$$

$$\equiv - \sum_{k=1}^{p} \sum_{r=1}^{p} l_{k,r}^{(i,j)} \frac{\partial v_j}{\partial x_k} \frac{\partial u_i}{\partial x_r} + l^{(i,j)} v_j u_i,$$

* See Irwin, loc. cit., page 11.

where

$$P_k^{(i,j)} \equiv \sum_{r=1}^{p} l_{k,r}^{(i,j)} v_j \frac{\partial u_i}{\partial x_r}; \quad Q_k^{(i,j)} \equiv \sum_{r=1}^{p} l_{k,r}^{(i,j)} u_j \frac{\partial v_i}{\partial x_r}, \quad i, j = 1, \cdots, n.$$

Thus we have

Theorem 4. If L be a self-adjoint system of second order linear partial differential expressions, there is a three term form of Lagrange's Identity:

$$\sum_{i=1}^{n} v_i L_i - \sum_{k=1}^{p} \frac{\partial P_k}{\partial x_k} \equiv \sum_{i=1}^{n} u_i M_i - \sum_{k=1}^{p} \frac{\partial Q_k}{\partial x_k}$$

$$\equiv -\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{r=1}^{p} \sum_{k=1}^{p} l_{k,r}^{(i,j)} \frac{\partial v_j}{\partial x_k} \frac{\partial u_i}{\partial x_r} + \sum_{i=1}^{n} \sum_{j=1}^{n} l^{(i,j)} u_i v_j,$$

where

$$P_k \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} P_k^{(i,j)} \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{r=1}^{p} l_{k,r}^{(i,j)} v_j \frac{\partial u_i}{\partial x_r},$$

$$Q_k \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} Q_k^{(i,j)} \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{r=1}^{p} l_{k,r}^{(i,j)} u_j \frac{\partial v_i}{\partial x_r},$$

the integration of which leads to a corresponding three term form of Green's Theorem.

Consider the transformation, T, defined by

$$u_j = \sum_{s=1}^{n} \psi_{j,s} \eta_s, \qquad |\psi_{1,1} \cdots \psi_{n,n}| \neq 0, \qquad j = 1, \cdots, n,$$

where the $\psi_{j,s}$ are functions of $x_1, \cdots, x_p$, which, together with all their partial derivatives of the first two orders and all their cross derivatives of the second order, are continuous in the region Ω. The functions η_s are to satisfy all the conditions imposed on the u_j. By this transformation the system L is carried over into the system Λ:

$$\Lambda_i(\eta_1, \cdots, \eta_n) \equiv \sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{r=1}^{p} \lambda_{k,r}^{(i,j)} \frac{\partial^2 \eta_j}{\partial x_k \partial x_r} + \sum_{j=1}^{n} \sum_{k=1}^{p} \lambda_k^{(i,j)} \frac{\partial \eta_j}{\partial x_k} + \sum_{j=1}^{n} \lambda^{(i,j)} \eta_j,$$

$$i = 1, \cdots, n.$$

The formulae for the new coefficients are*

$$\lambda_{k,r}^{(i,j)} \equiv \sum_{s=1}^{n} l_{k,r}^{(i,s)} \psi_{s,j},$$

$$\lambda_{k}^{(i,j)} \equiv 2 \sum_{s=1}^{n} \sum_{r=1}^{p} l_{k,r}^{(i,s)} \frac{\partial \psi_{s,j}}{\partial x_r} + \sum_{s=1}^{n} l_{k}^{(i,s)} \psi_{s,j},$$

$$\lambda^{(i,j)} \equiv \sum_{s=1}^{n} \sum_{k=1}^{p} \sum_{r=1}^{p} l_{k,r}^{(i,s)} \frac{\partial^2 \psi_{s,j}}{\partial x_k \partial x_r} + \sum_{s=1}^{n} \sum_{k=1}^{p} l_{k}^{(i,s)} \frac{\partial \psi_{s,j}}{\partial x_k} + \sum_{s=1}^{n} l^{(i,s)} \psi_{s,j},$$

$$i = 1, \cdots, n.$$

Definition. A set of functions, I, of the coefficients of a given system of differential expressions, L, and their derivatives, shall be called a *relatively invariant set of degree* μ, with respect to the transformation T, if every one of a similar set of functions, built up from the coefficients of the transform Λ and their derivatives, is expressible, by means of the above relations, as a linear combination of the functions of the original set, with coefficients of the form $\prod_{i,j} \psi_{i,j}^{a_{i,j}}$, where $\sum_{i,j} a_{i,j} = \mu$. If the coefficients in the combination are all unity, I shall be called an *absolutely invariant set*.

It will be found convenient to refer to the functions in I as the *elements* of the set.

If the transformation T be applied to Lagrange's Identity, the latter goes over into

$$\sum_{i=1}^{n} \left\{ v_i \Lambda_i - \eta_i \sum_{s=1}^{n} \psi_{s,i} M_s \right\} \equiv \sum_{k=1}^{p} \frac{\partial \overline{R}_k}{\partial x_k},$$

where the $\overline{R}_k$ are bilinear in the v_i and η_i and their partial derivatives of the first order. Hence Λ_i and

$$\sum_{s=1}^{n} \psi_{s,i} M_s, \qquad i = 1, \cdots, n,$$

are adjoint systems.

Theorem 5. If the dependent variables u_j, $j = 1 \cdots, n$, in the system of second order linear partial differential equations, L, be subjected to the transformation T, defined by the equations

$$u_j = \sum_{s=1}^{n} \psi_{j,s}(x_1, \cdots, x_p) \eta_s, \qquad j = 1, \cdots, n,$$

* Compare Irwin, loc. cit., page 18.

the expressions in $\mathfrak{M}$, the system adjoint to Λ, the transform of L under the transformation T, are linear combinations of the expressions in M, the system adjoint to L, the coefficients of the combinations being the $\psi_{j,s}$; — that is

$$\mathfrak{M}_i \equiv \sum_{s=1}^{n} \psi_{s,i} M_s, \qquad i = 1, \cdots, n.$$

It is at once evident that the coefficients of the system M, adjoint to the given system L, are the elements of a relatively invariant set of degree one under the transformation T.

Definition. Two configurations are said to be equivalent with respect to a given set of transformations, if there is one transformation of the set which carries the first configuration over into the second, and another which carries the second into the first.

Definition. Given two configurations, C and C', whose corresponding relatively invariant sets under a set of transformations T are I and I', then, if there be a set of functions Θ, of the independent variables, such that when the elements of I' are linear combinations of the elements of I with coefficients taken from Θ, the configurations C and C' are equivalent, the invariant set I is said to be *complete*.

In the case of absolute invariants all the elements of Θ are unity.

Let the configurations be the systems of second order linear partial differential expressions

$$L_i\, u(_1, \cdots, u_n)\,; \qquad \Lambda_i\,(\eta_1, \cdots, \eta_n), \quad i = 1, \cdots, n,$$

whose adjoint systems are, respectively,

$$M_i\,(v_1, \cdots, v_n)\,; \quad \mathfrak{M}_i\,(v_1, \cdots, v_n), \qquad i = 1, \cdots, n.$$

Let Θ be composed of the functions

$$\theta_{i,j}\,(x_1, \cdots, x_p), \quad i,j = 1, \cdots, n.$$

Now a known invariant set of degree one for either configuration under a transformation T of the form

$$u_i = \sum_{j=1}^{n} \psi_{i,j}\,\eta_j\,; \qquad |\,\psi_{1,1} \cdots, \quad \psi_{n,n}\,| \neq 0 \;; \quad i = 1, \cdots, n,$$

has for its elements the coefficients of the adjoint system. Suppose the coefficients of $\mathfrak{M}$ to be linear combinations of the corresponding coefficients of M, the coefficients of the combination being taken from Θ so that

$$\mathfrak{M}_i \equiv \sum_{j=1}^{n} \theta_{i,j} M_j, \qquad i = 1, \cdots, n.$$

Subject L to the transformation

$$u_i = \sum_{j=1}^{n} \theta_{i,j} \eta_j, \qquad i = 1, \cdots, n,$$

and assume the $\theta_{i,j}$ to have been so chosen as to satisfy the relation $|\theta_{1,1}, \cdots, \theta_{n,n}| \neq 0$. The transformation is then of the form T, and L is carried over into $\bar{\Lambda}$, whose adjoint is

$$\sum_{j=1}^{n} \theta_{i,j} M_j \equiv \mathfrak{M}_i, \quad i = 1, \cdots, n.$$

But $\bar{\Lambda}$ and Λ, having the same adjoint, are identical, hence there is one transformation of the form T which carries L into Λ. The inverse transformation exists and is also of the form T. In a similar fashion it may be proved to carry Λ into L. Hence:

Theorem 6. The coefficients of a system of second order linear partial differential expressions M, adjoint to a given system L, are the elements of a relatively invariant set of degree one under any transformation of the form T, and this relatively invariant set is complete.

Definition. By the partial weight of any coefficient of the system L with respect to any one of the independent variables, shall be meant the number of lower indices which refer to that variable. The $\psi_{i,j}$ shall all be of weight zero with respect to each, that is every one, of the independent variables. To find the partial weight of a partial derivative of one of the above functions with respect to any one of the independent variables, diminish the partial weight of the function itself with respect to the variable in question by the number of times it has been differentiated with respect to that variable. The total weight of any of the above functions shall be the sum of its partial weights. The weight (total or partial) of a product shall be equal to the sum of the weights (total or partial) of its factors. A polynomial is isobaric, totally or partially, if all of its terms are of the same total or partial weight.

With this convention as to weights, an obvious extension of the

methods employed by Dr. Irwin* to establish his propositions 5 and 12 leads to the following theorems.

Theorem 7. An element of an invariant set may or may not be isobaric; but if it is not, it is merely the sum of elements which are.

This theorem applies to the total weight or to any one of the partial weights.

Theorem 8. An element of an invariant set of functions built up from the coefficients of a system of second order linear partial differential expressions, adjoint to a given system L, if not itself homogeneous in each and every coefficient present and its derivatives, is merely the sum of elements which are.

Theorem 9. An element of an invariant set of functions built up from the coefficients of the adjoint system, if it is homogeneous of degrees ν in some one coefficient and its derivatives, and contains no other coefficient, is essentially nothing but the νth power of the coefficient present.

By means of theorems 7, 8 and 9 may be proved our final theorem.

Theorem 10. Essentially the only relatively invariant set of degree one for the system L under the transformation T, has for its elements the coefficients of the system adjoint to L.

Under the special transformation $u_i = \psi_i \eta_i$, $i = 1, \cdots, n$, the invariant set becomes a set of invariant functions* to which Irwin's theorems apply almost verbatim.

In addition to the article in the *Encyklopädie*, to which reference has already been made, special cases of certain theorems in this paper are developed by Professor Max Mason in his article on "Green's Theorems and Green's Functions for Certain Systems of Differential Equations",† and by Jordan in his *Cours d'Analyse* (ed. 1887, tome 3, p. 142, § 115).

BROOKLYN, N. Y.,
MAY, 1910.

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* Loc. cit., p. 19, and pp. 25, 26.

*See Irwin, loc. cit., p. 18.

† *Trans. Am. Math. Soc.*, vol. 5 (1904), p. 220.

SOME CIRCLES ASSOCIATED WITH CONCYCLIC POINTS

By J. L. Coolidge

If a system of any number of lines, n, be given in a plane, they may be associated with n circles or n points in a variety of ways. The first of these was discovered by Clifford, and gives rise to the following theorem: *

Let n lines be given in a plane, no two of which are parallel. If n be even we may associate with them a point, and if n be odd a circle, in such a way that the point is common to the n circles each associated with a set of $n-1$ lines obtained by omitting each of the given lines in turn; the circle contains the n points each associated with a set of $n-1$ lines in the same way.

A more remarkable theorem of the same sort was discovered in 1891 by Pesci; it may be stated as follows: †

Let n lines be given in a plane, no two of which are parallel. We may associate with them a point and a circle in such a way that the point is common to the n circles each associated with a set of $n-1$ lines obtained by omitting each of the given lines in turn, while the circle contains the centres of these n circles. When $n > 4$ the point will not lie on the circle.

In the special case, $n = 4$, the circles are those circumscribed to the four triangles formed by the four lines. These circles pass through the focus of the parabola which touches the four lines, and their centres lie on a circle through that focus. This theorem is due to Steiner. ‡

There is one more theorem of this same sort which was given by Grace, and may be stated as follows: §

Let n lines be given in a plane, no two of which are parallel, and n concyclic points, one on each of them. If n be odd we may associate with them a point, and if n be even a circle, in such a way that the point is common to the n circles each associated with a set of $n-1$ lines obtained by omitting each of the given lines in turn; the circle contains the n points each associated with a set of $n-1$ lines in the same way.

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* A synthetic Proof of Miquel's Theorem, *Oxford, Cambridge and Dublin Messenger of Mathematics*, vol. 5, 1870. At least two subsequent writers seem to have rediscovered it.

† Dei cercoli circonscritti ai triangoli formati di n rette in un piano: *Periodico di Matematica*, vol. 5, 1891.

‡ See his *Collected Works*, vol. 1, p. 223.

§ Circles, Spheres and Linear Complexes, *Transactions of the Cambridge Philosophical Society*, vol. 16, 1898.

When the circle of the n points reduces to a line, Grace's theorem reduces to Clifford's.

The first theorem which I wish to develop in the present paper is analogous to both Pesci's and Grace's, and may be stated as follows:*

I) *Given n points on a circle where $n \geqq 4$. We may associate with them a point and a circle in the following manner:*

a) *the point is the centre of the circle;*

b) *the radius of the circle is one half that of the given circle;*

c) *the point lies on the n circles each associated with a system of $n - 1$ concyclic points obtained by omitting each of the given points in turn;*

d) *the circle contains the centres of these n circles.*

The method of proof is substantially that which is used to advantage for Clifford and Grace's theorems, and depends upon the two following simple theorems:

1) *If a triangle be given, and a point be marked on each side, the three circles, each of which passes through a vertex and the two marked points of the adjacent sides are concurrent.*†

We see, in fact, that each circle and the opposite side of the triangle constitute a cubic curve, and as these three cubics have eight common points, they have a ninth point in common also. The theorem is also easily proved by elementary geometry. If we invert this figure with regard to any convenient circle, we shall get

2) *Given a circular triangle formed by three circles through a common point. If a point be marked on each side of the triangle, the three circles, each of which passes through a vertex and the two marked points of the adjacent sides, are concurrent.*

Let us give one corollary to the last theorem which will be of use to us in the present paper, though not for our immediate purpose:

3) *Given four points upon a circle arranged in cyclic order, and four circles of arbitrary radius each connecting a successive pair of points. Then the remaining intersections of successive pairs of circles lie on another circle.*‡

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* Since the present article went to press I have found the analog of this theorem in three dimensions, though not in so many words in an article by Intrigila, "Sul tetraedro." *Rend. Accad. Sci. di Napoli*, vol. 22, 1883. It is the more remarkable that the present theorem should seem to be new.

† Miquel, Théorèmes de Géometrie, *Liouville's Journal*, vol. 3, 1838.

‡ Miquel, "Théorèms de Géometrie," *Liouville's Journal*, vol. 9, 1844.

The four points shall be called $P_1\ P_2\ P_3\ P_4$, the circle upon which they lie c, while c_{ij} is the circle through P_i and P_j. The circles c_{ij} and c_{jk} shall intersect in P_j and P'_j.

The circles $c\ c_{34}\ c_{41}$ are concurrent in P_4. They form a circular triangle whose vertices are $P_1\ P_3\ P_4$. The side c contains also P_2, the side c_{34} contains P'_3, and the side C_{41} contains P'_1. On the other hand $P_1\ P'_1\ P_2$ lie upon c_{12} and $P_3\ P'_3\ P_2$ lie upon c_{23}. These intersect again in P'_2; hence, by 2) P'_2 lies on the circle through $P'_4\ P'_1\ P'_3$.

We may now proceed to prove our Theorem I:

When $n = 2$ we shall associate with the given points the point mid way between them. When $n = 3$ we shall associate with the given points their nine-point circle and its centre. Notice that this circle passes through the three points associated with each pair of the given points. We may, without restriction, assume that the radius of the circumscribed circle is equal to 2. When $n = 4$ we have a well-known theorem whereby the nine-point circles of the four triangles formed by four concyclic points are concurrent.*

We see, in fact, that all conics of the pencil through the vertices and ortho-centre of a triangle must be rectangular hyperbolas, for the involution which they determine on the line at infinity has three mutually perpendicular directions. The locus of the centres of these conics is a conic through the double points of this involution, which are the circular points at infinity, and the vertices of the common self-conjugate triangle, which are the feet of the altitudes. In other words, the nine-point circle is the locus of the centres of all conics through the three vertices of a triangle and the ortho-centre, and if four points be given, which are not the vertices and ortho-centre of a triangle, the nine-point circles of their four triangles are concurrent in the centre of the rectangular hyperbola through the four points. When, further, the points are concyclic, their four nine-point circles have all the same radius, and so their centres lie on a circle of equal radius around the point of concurrence as centre. Our theorem is thus established in the case $n = 4$.

Let us pass on to the case $n = 5$. Here we shall have five concyclic points $P_1\ P_2\ P_3\ P_4\ P_5$.

The circle associated with $P_j\ P_k\ P_l\ P_m$ shall be called c_i, its centre C_i. The nine-point circle of $P_k\ P_l\ P_m$ shall be c_{ij}, its centre C_{ij}. The middle point of $P_l\ P_m$ shall be P_{ijk}. It appears then that $P_{ijk}\ P_{ikl}\ P_{kli}\ P_{lij}$ lie

* Perhaps the first proof of this was given by Greiner: Ueber das Kreisviereck, *Grunert's Archiv der Math. und Phys.*, vol. 60, 1877.

upon a circle of radius 1 whose centre lies half-way from P_m to the centre of the circumscribed circle Q_m. The points $P_{ij}\,P_{jk}\,P_{ki}$ are all at a distance unity from P_{ijk} and so lie on a circle of radius 1 through Q_m. This circle may, for the present be called c_{ijk}. Consider, now, the three circles $c_i\,c_j\,c_k$.

c_i contains C_{ij}, C_{ki}, C_{il};
c_j contains C_{ij}, C_{jk}, C_{jl};
c_k contains C_{jk}, C_{ki}, C_{kl}.

But C_{il}, C_{jl}, C_{k}, are the vertices of a circular triangle, with the three following sides:

c_{ijl} containing C_{il}, C_{ij}, C_{jl};
c_{jkl} containing C_{jl}, C_{jk}, C_{kl};
c_{kil} containing C_{kl}, C_{ki}, C_{il};

and the three circles $c_{ijl}\,c_{jkl}\,c_{kil}$ are concurrent in Q_m. Hence, by Theorem 2 the three circles c_i, c_j, c_k are concurrent, or all five circles c_i pass through a point. But all have the radius 1. Hence their centres lie on a circle of radius 1 about the point of concurrence as centre.

We have thus proved our theorem for $n = 4$ and $n = 5$. But this very proof is independent of the number 5, and shows equally well that if the theorem hold for $n - 2$ and $n - 1$ it holds for n also, the only modification being that c_{ijk} has now a natural instead of an artificial significance, and Q_m is replaced by P_{ijkl}, a definite point in every case where $n \geqq 6$. Our theorem is thus proved in its entirety.

We may easily obtain other systems of circles, similarly connected, by a dilatation away from the centre of the circumscribed circle with any chosen ratio. For instance if we dilate with the ratio 2, in the case where $n = 4$, we pass from the circle through the centres of four nine-point circles to that through four orthocentres, for the orthocentre is twice as far away from the centre of the circumscribed circle as is the centre of the nine-point circle.

It is now time to redeem our pledge to make some use of Theorem 3. Suppose that we have four concyclic points $P_1\,P_2\,P_3\,P_4$. They may be arranged in three different cyclic orders. The circle through P_i and P_j shall be called c_{ij}. The second intersection of c_{ij} and c_{jk} shall be $P_{ij,jk}$. We have twelve such points arranged as follows:

$P_{14,42}$, $P_{42,23}$, $P_{23,31}$, $P_{31,14}$ lying on c_2',
$P_{12,23}$, $P_{13,34}$, $P_{34,41}$, $P_{41,12}$ lying on c_3',
$P_{13,34}$, $P_{34,42}$, $P_{42,21}$, $P_{21,13}$ lying on c_4'.

Consider the sextic curve $(c_2' c_{12} c_{34})$. It passes through each ordinary intersection of the quartics $(c_{13} c_{24})$ and $(c_{14} c_{23})$, and has at each circular point at infinity a triple point, whereas each quartic has a double point there. We must then by Nöther's fundamental theorem have an identity of the sort

$$(c_2' c_{12} c_{34}) \equiv \phi_3(c_{13} c_{24}) + \phi_4(c_{14} c_{23}).$$

The curves ϕ_3 and ϕ_4 must be conics and each must pass once through each circular point. Hence they are circles.

Moreover ϕ_3 contains $P_{12,23}$, $P_{23,34}$, $P_{34,41}$, $P_{41,12}$ and so must be identical with c_3', and ϕ_4 for a like reason is identical with c_4'. Hence finally, the points common to c_3' and c_4' belong to c_2' also, that is to say:

II) *Four concyclic points may be arranged in three different cyclic orders, each giving rise to a cyclic order among four out of six circles of arbitrary size, which connect the given points in pairs. Successive circles in each cyclic order will intersect again in four concyclic points, and the three circles so determined are co-axial.*

It is natural to ask what analogues exist in three dimensions to the various theorems which we have here discussed. There are two rival claimants to the honor to be the three dimensional analogue of the nine-point circle, but neither seems to give rise to anything corresponding to Theorem 2. Theorem 1, on the other hand, may easily be generalized as follows:*

4) *Let a point be marked on each edge of a tetraedron, and let a sphere be passed through each vertex, and the marked points of the adjacent edges; these four spheres will be concurrent.*

This theorem may be proved in much the same way as was 1. There is an analogue to 3) also which has not, so far as I know, ever been published.

III) *Given five points upon a sphere arranged in cyclic order. Through each set of three successive points a sphere of arbitrary size shall be passed, thus determining five other points, each the remaining intersection of three successive spheres. These five points also lie upon a sphere.*

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*This theorem was first proved by Roberts: On certain Tetraedra specially related to four Spheres meeting in a point, *Proceedings London Mathematical Society*, vol. 11, 1880. It was given implicitly much earlier by Miquel: Memoire de Geometrie, *Liouville's Journal*, vol. 11, 1845.

It is remarkable that whereas Theorem 3 comes immediately from Theorem 1 by an inversion, its analogue III does not seem to bear any close relation to 4. We find a proof on different lines as follows: The five given points shall be P_1, P_2, P_3, P_4, P_5, lying on s. The sphere through $P_i P_j P_k$ shall be called s_{ijk}; the remaining intersection of $s_{ijk} s_{jkl} s_{klm}$ shall be P'_k. Let us consider the following equation:

$$\lambda_1 s_{451} s_{123} + \lambda_2 s_{512} s_{234} + \lambda_3 s_{123} s_{345} + \lambda_4 s_{234} s_{451} + \lambda_5 s_{345} s_{512} = 0.$$

The quantities λ_i being constants, we see that we have here a cyclid through all ten of our points P_i P_i'. There is no linear dependence among the various terms, so that by varying the λ_i's we have really a four parameter family of surfaces. They will intersect s in a system of cyclics through five fixed points. Let us make use of two of our parameters to pass these curves through two other chosen points. We have still two free parameters, whereas the cyclics on any sphere through seven points contain an eighth point also, and form a one-parameter system, as we easily see by projecting them into cubics from one of their intersections. Hence, we may use our remaining parameters to make the cyclid include the sphere s entirely. The remainder of the cyclid will be a sphere through the five points P_i'.

There does not seem to be any simple three-dimensional analogue to II.

Cambridge, Mass.,
January, 1910.

ON A METHOD FOR THE SUMMATION OF SERIES

By Rutherford E. Gleason

If r is an integer and if for values of x greater than r, $f(x) > 0$, $f'(x) < 0, f''(x) > 0$, and $\int_r^\infty f(x)dx$ is finite and determinate, then

$$\sum_{n=r}^{n=\infty} f(n)$$

will be a convergent series of positive terms each less than the one before it. The graph of $f(x)$, for values of x greater than r, will be a curve convex downward and having the axis of X as an asymptote. If we erect ordinates corresponding to $x = r$, $x = r + 1$, $x = r + 2$, etc., draw the chords of the arcs into which they divide the graph and draw through their extremities lines parallel to the x axis as in the figure, we see that the sum of the rectangles is

$$\sum_{r+1}^{\infty} f(n),$$

the sum of the triangles is $\frac{1}{2} f(r)$; and that the sum of the segments bounded by the chord and arc is less, usually very much less, than the sum of the triangles and therefore than $\frac{1}{2}f(r)$. Let the sum of the segments which is a new convergent series consisting of very much smaller terms than the original series be ϵ. Now the sum of the rectangles plus the sum of the triangles minus the sum of the segments is the area bounded by the curve, the ordinate corresponding to $x = r$ and the asymptote, *i. e.*, $\int_r^\infty f(x)dx$, and we have

$$\sum_{r+1}^{\infty} f(n) + \tfrac{1}{2} f(r) - \epsilon = \int_r^\infty f(x)\,dx$$

or

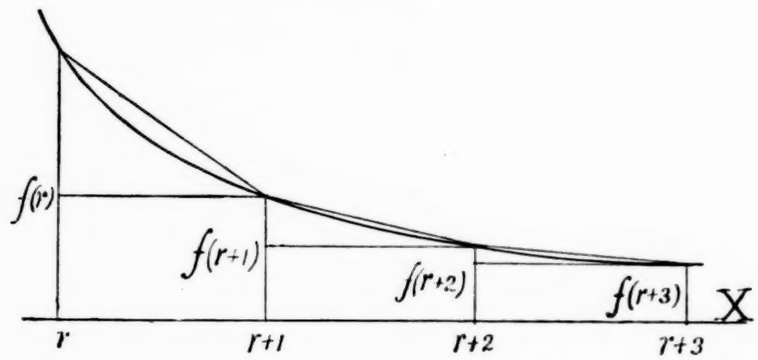
$$(1) \qquad \sum_{r}^{\infty} f(n) = \int_r^\infty f(x)\,dx + \tfrac{1}{2} f(r) + \epsilon = \mu + \epsilon$$

Call any term of ϵ, *i. e.*, any segment in the figure, $f_1(n)$; then

$$(2) \qquad f_1(n) = \tfrac{1}{2}\Big[f(n) + f(n+1)\Big] - \int_n^{n+1} f(x)\,dx$$

and

$$\epsilon = \sum_r^{\infty} f_1(n).$$



If $f_1(x)$ satisfies the conditions we imposed upon $f(x)$ we may treat the new series as we treated the original one:

$$\epsilon = \sum_r^{\infty} f_1(n) = \int_r^{\infty} f_1(x)\,dx + \tfrac{1}{2} f_1(r) + \epsilon_1 = \mu_1 + \epsilon_1 \text{ and so on.}$$

We shall have the following formulæ:

$$f_m(n) = \frac{1}{2}\Big[f_{m-1}(n) + f_{m-1}(n+1)\Big] - \int_n^{n+1} f_{m-1}(x)\,dx, \qquad \text{(I)}$$

$$\epsilon_{m-1} = \sum_r^{\infty} f_m(n) = \int_r^{\infty} f_m(x)\,dx + \tfrac{1}{2} f_m(r) + \epsilon_m = \mu_m + \epsilon_m, \qquad \text{(II)}$$

$$\sum_r^{\infty} f(n) = \sum_0^{m} \mu_k + \epsilon_m. \qquad \text{(III)}$$

The new series $\Sigma \mu_k$ by which we replace $\Sigma f(n)$ is easily seen to converge very rapidly.

There now remains but to find methods for approximating to ϵ_m. If the ratio of the segment to the triangle in the curve $y = f_m(x)$ decreases as x increases, a very excellent approximation is found in

$$f_{m+1}(r) < \sum_{r}^{\infty} f_{m+1}(n) < \frac{f_m(r) \cdot f_{m+1}(r)}{f_m(r) - f_m(r+1)}. \qquad \text{(IV)}$$

The proof of IV is as follows: if the ratio of the segment to the triangle is decreasing, the ratio of the first segment to the first triangle is greater than the ratio of the sum of the segments to the sum of the triangles, *i. e.*,

$$f_{m+1}(r) \ : \ \tfrac{1}{2}[f_m(r) - f_m(r+1)] > \epsilon_m \ : \ \tfrac{1}{2} f_m(r).$$

When IV fails, another and very useful formula, but not so accurate as the former, when that can be used, is always available:

$$\sum_{n}^{\infty} f_{m+1}(n) < \tfrac{1}{2}[f_m(r) - f_m(r+1)]. \qquad \text{(V)}$$

This is easily proved.—Continue the chord connecting the $(n+1)$th and $(n+2)$th ordinates, in the graph of $f_m(x)$, to the left to the nth ordinate. The area included between this continuation of the chord, the nth ordinate and the chord above is $\tfrac{1}{2}[f_m(n) - 2f_m(n+1) + f_m(n+2)]$, which is greater than the included segment. The sum from r to ∞ of the areas thus formed is $\tfrac{1}{2}[f_m(r) - f_m(r+1)]$.

As an example, the method will be applied to the summation of $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ etc. It is found convenient to drop the first term and combine the alternate positive and negative terms in pairs and reverse the signs, so that we may place

$$\sum f(n) = \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{7} - \frac{1}{9}\right) + \left(\frac{1}{11} - \frac{1}{13}\right) \cdots.$$

Let the summation be commenced with the sixth term of this series.

We have

$$f(x) = \frac{1}{4x-1} - \frac{1}{4x+1}$$

$$\int_6^\infty f(x)dx = \tfrac{1}{4}\log\frac{25}{23} = \ .020845405$$

$$\tfrac{1}{2}f(6) = \frac{1}{46} - \frac{1}{50} = \ \underline{.001739130}$$

$$\therefore \mu_0 = \ .022584535$$

$$f_1(x) = \frac{1}{2}\left(\frac{1}{4x-1} - \frac{1}{4x+1} + \frac{1}{4x+3} - \frac{1}{4x+5}\right) - \int_x^{1+x}\left(\frac{1}{4x-1} - \frac{1}{4x+1}\right)dx$$

$$\int_6^\infty f_1(x)dx = \frac{1}{16}\Big(-25\log 23 + 27\log 25 + 25\log 27 - 27\log 29\Big) = .000076632$$

$$\frac{1}{2}f_1(6) = \frac{1}{4}\Big(\frac{1}{23} - \frac{1}{25} + \frac{1}{27} - \frac{1}{29}\Big) + \frac{1}{8}\Big(\log 23 - \log 25 - \log 27 + \log 29\Big) = \underline{.000017805}$$

$$\therefore \mu_1 = .000094437$$

We shall now find the limit of error in placing $\mu_0 + \mu_1 = \sum_6^\infty f(n)$.

The ratio

$$\frac{\frac{1}{2}\Big[f_1(x) + f_1\,(1+x)\Big] - \int_x^{1+x} f_1(x)dx}{\frac{1}{2}\,[f_1(x) - f_1(1+x)]}$$

decreases as x increases so that IV may be here applied. We find

$$f_2(6) = .000000934$$

Substituting this in IV we get

$$.00000\,0934 < \sum_6^\infty f_2(n) < .00000\,2131.$$

We also have

$$\sum_1^5 f(n) = .191921048$$

$$\mu_0 = .022584535$$

$$\mu_1 = \underline{.000094437}$$

$$.214600020$$

Taking into account the limits established by IV, we find

$$.214600954 < \sum_1^\infty f(n) < .214602151$$

which subtracted from unity places the sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.} \left(= \frac{\pi}{4}\right)$ between $.78539\,7849$ and $.78539\,9046$. Although this is but a crude example, it is an illustration of how this method may be applied. It is seen that two terms of $\Sigma\,\mu_k$ are here equivalent to hundreds in the original series.

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RATIONALITY GROUPS IN PRESCRIBED DOMAINS

By S. Epsteen

1. Introduction. Just as the galoisian group of an algebraic equation reflects the essential characteristics of the equation, so does the rationality group reflect the essential characteristics of a linear homogeneous differential equation.

In studying an algebraic equation

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

by the Galois' method, we first specify a domain of rationality, and a certain number of the $n!$ permutations of the roots $x_1, \cdots x_n$, constitute the galoisian group of the equation.

Domain of rationality. A domain of rationality is a set of numbers whose sum, difference, product, and quotient (exclusive of the zero divisor) always give a number of the set. Frequently, the domain of rational numbers, r, is taken. One also uses the domain $c = r[i]$, that is, the domain r with $i = \sqrt{-1}$ adjoined; this gives numbers of the form $u + iv$, u and v denoting numbers of r. Likewise, the domain of real numbers, R, and the domain $C = R[i]$, are used. In some instances, when one wishes to state a theorem for any arbitrary domain, the symbol D is convenient.

Galois' group. The group G of an equation in a given domain D is the totality of substitutions having the following double property:

a) Every rational function of the roots, numerically unaltered by all the substitutions of the group, has its numerical value in D; and

b) Every rational function of the roots, having its numerical value in D, is numerically unaltered by the substitutions of G.

In studying a linear homogeneous differential equation

$$K(y) \equiv \frac{d^n y}{dx^n} + k_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + k_n(x) \cdot y = 0$$

by the Picard-Vessiot method, we first specify a derivative domain of ration-

ality, and a certain sub-group of the general linear homogeneous group in n variables and n^2 parameters

$$L_{n^2}: \quad \eta_{j_1} = \sum_{j_2} a_{j_1 j_2} y_{j_2} \qquad (j = 1 \cdots n)$$

constitutes the rationality group of the equation. Here $y_1, \cdots y_n$ denote a fundamental system of K, and $|a_{j_1 j_2}| \neq 0$.

Derivative domain of rationality. A derivative domain of rationality is composed of one of the foregoing domains r, c, R, C, and any arbitrary number of functions analytic in a specified region of the plane of the complex variable x. The operations of addition, subtraction, multiplication, division (exclusive of the divisor zero), and *differentiation* always reproduce elements of the domain. We denote by $R[x]$ and $C[x]$ the domains R and C with the independent variable x adjoined: * by $R[x, f(x)]$ the domain $R[x]$ with $f(x)$ adjoined. In some instances, when one wishes to state a theorem for any arbitrary domain, the symbol $D\{x\}$ is convenient.

A domain $D\{x\}$ being given, a number or a function is said to be *rational* if it is in this domain, and *irrational* if it is not in this domain.

Group of Rationality. By the rationality group G of a linear homogeneous differential equation of order n, in a domain $D\{x\}$, is meant the totality of linear homogeneous substitutions of non-vanishing determinant on a fundamental system of integrals $y_1, \cdots y_n$:

$$\eta_{j_1} = \sum_{j_2} a_{j_1 j_2} y_{j_2}, \qquad (j = 1 \cdots n)$$

which possess the property known as the Picard-Vessiot Double Theorem: †

A) Every rational differential function of $y_1, \cdots y_n$:

$$F(x, y_1 \cdots y_n, y_1' \cdots y_n', \cdots)$$

which is numerically invariant under the substitutions of G has its numerical value, $r(x)$, in the domain $D\{x\}$; and inversely,

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* The necessity of explicitly adjoining the independent variable x was first pointed out by A. Loewy, *Mathematische Annalen*, vol. 59, p. 436. A domain may contain functions of x without containing x itself, as for example $R[e^x]$.

† Schlesinger, *Handbuch der Theorie der linearen Differentialgleichungen*, II, 1, p. 71.

B) Every rational differential function of $y_1, \cdots y_n$:

$$F(x, y_1 \cdots y_n, \; y_1' \cdots y_n', \cdots)$$

which has a numerical value $r(x)$ in the domain $D\{x\}$ is invariant under the substitutions of G.

The literature of the Picard-Vessiot Theory is now quite extensive, but any one familiar with the elements of the Galois' Theory of Algebraic Equations and Lie's Theory of Continuous Groups will find it easy to read the summaries in Picard's *Traité d'Analyse*, vol. 3, last chapter; Schlesinger's *Handbuch der Theorie der linearen Differentialgleichungen*, II, 1, pp. 1–125; and Klein's *Höhere Geometrie*, vol. 2, pp. 265–302.

2. The Equations and Domains. In the *Vorlesungen über Höhere Geometrie*, vol. 2, p. 300, Klein points out the desirability of giving examples of differential equations with their rationality groups for specified domains of rationality. In the present paper this suggestion is carried out for the following equations and domains:

I. Equation of the Second Order, with constant coefficients:

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0,$$

for the domains of rationality:

a) R; *b*) C; *c*) $R[x]$; *d*) $C[x]$; *e*) $R[x, e^x]$; *f*) $C[x, e^x] \equiv C[x, \sin x]$.

II. Hypergeometric (Gauss') equation:

$$\frac{d^2y}{dx^2} + \frac{[\gamma - (\alpha + \beta + 1)x]}{x(1-x)} \frac{dy}{dx} - \frac{\alpha\beta}{x(1-x)} y = 0,$$

for the domains $R[x]$ and $C[x]$.

III. Moduli of Periodicity of Elliptic Integrals of the First Kind:

$$\frac{d^2y}{dx^2} + \frac{1-2x}{x(1-x)} \frac{dy}{dx} - \frac{1}{4x(1-x)} y = 0,$$

for the domains $R[x]$ and $C[x]$.

IV. Legendre's Polynomial of order k:

$$\frac{d^2y}{dx^2} + \frac{1-2x}{x(1-x)}\frac{dy}{dx} + \frac{k(k+1)}{x(1-x)}y = 0 \quad (k = \text{positive integer});$$

domains $R[x]$, $C[x]$.

V. Bessel's Equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0;$$

domains: *a*) $R[x]$; *b*) $C[x]$; *c*) $R[x, \sin x]$; *d*) $C[x, \sin x]$.

VI. Cauchy's Equation:

$$\frac{d^2y}{dx^2} + \frac{a_1}{x}\frac{dy}{dx} + \frac{a_2}{x^2}y = 0;\text{*}$$

domains: *a*) R; *b*) C; *c*) $R[x]$; *d*) $C[x]$; *e*) $C[x, \log x]$.

The solution of I, *d*, has been given, in the main, by Vessiot.† All the other examples I believe to be new.

3. Algebraic Subgroups of L_{2^2}. The equations investigated in this paper are all of the second order. Following the outline of Vessiot‡ we write down the algebraic subgroups of the general linear homogeneous group L_{2^2} in 2 variables and 2^2 parameters,§ whose infinitesimal transformations are

$$L_2^2: \qquad y_1p_1, \quad y_1p_2, \quad y_2p_1, \quad y_2p_2; \qquad \left(p_i \equiv \frac{\partial f}{\partial y_i}\right),$$

together with their differential invariants of first order.

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* Our results apply equally well to Cauchy's equation in the form

$$y'' + \frac{a_1}{x-a}y' + \frac{a_2}{(x-a)^2}y = 0.$$

† E. Vessiot, *Annales de l'Ecole Normale Supérieure*, vol. 9 (1892), pp. 248–9. Vessiot found the group of the equation of order n with constant coefficients, but did not specify a domain of rationality. Applied to $n = 2$, his determination is that of §5 *d*; cases 1 and 3.

‡ Vessiot, loc. cit., p. 249. F. Marotte, *Comptes Rendus*, vol. 124 (1897), pp. 608–610.

§ Vessiot, loc. cit., p. 258.

GROUP		DIFFERENTIAL INVARIANTS
A.	$y_1p_2,\ y_1p_1 - y_2p_2,\ y_2p_1$ (special linear group)	$y_1y_2' - y_1'y_2 \equiv \Delta$
B.	$y_1p_1,\ y_1p_2,\ y_2p_2$	$\dfrac{y_1'}{y_1}$
C.	$y_1p_2,\ y_1p_1 + y_2p_2$	$\dfrac{y_1'}{y_1},\ \dfrac{\Delta}{y_1^2} \equiv \dfrac{d}{dx}\left(\dfrac{y_2}{y_1}\right) \equiv w$
D.	$y_1p_1,\ y_2p_2$	$\dfrac{y_1'}{y_1},\ \dfrac{y_2'}{y_2}.$
E.	$ay_1p_1 + by_2p_2,\ y_1p_2$	$\dfrac{y_1'}{y_1},\ \dfrac{\Delta^a}{y_1^b}$
F.	$y_1p_1 + y_2p_2$	$\dfrac{y_1'}{y_1},\ \dfrac{y_2'}{y_2},\ \dfrac{y_2}{y_1} \equiv v$
G.	y_1p_2	$y_1,\ \Delta$
H.	$ay_1p_1 + by_2p_2$	$\dfrac{y_1'}{y_1},\ \dfrac{y_2'}{y_2},\ \dfrac{y_2^a}{y_1^b}.$

Since the listed invariants must be rational, the parameters a and b in E and H are necessarily integral numbers.

4. The Resolvents.* For an equation of second order,

$$K(y) = \frac{d^2y}{dx^2} + k_1(x)\frac{dy}{dx} + k_2(x)y = 0,$$

the invariant Δ satisfies the resolvent

$$R_A: \qquad \Delta' + k_1\Delta = 0.$$

It is best to replace the invariant y_1'/y_1 of the groups B, C, D, E, F, (G), H by

$$u_1 = \frac{y_1'}{y_1} - \frac{1}{2}\,\frac{y_1y_2'' - y''y_2}{y_1y_2' - y_1'y_2} = \frac{y_1'}{y_1} + \frac{1}{2}k_1.$$

* Vessiot, loc. cit., p. 217 calls this the *transformed*. I prefer the term *resolvent* by analogy with the Galois' Theory of Equations.

This function satisfies the resolvent Riccati equation

$$R_B: \qquad u' + u^2 + I = 0,$$

where

$$I = k_2 - \frac{1}{4}k_1^2 - \frac{1}{2}k_1'.$$

The invariant $v = y_2/y_1$ of the group F satisfies the resolvent

$$R_F: \qquad \left(\frac{-v''}{2v'}\right)' + \left(\frac{-v''}{2v'}\right)^2 + I = 0.$$

From the latter it is easy to see that the invariant

$$w = \frac{d}{dx}\left(\frac{y_2}{y_1}\right) = v'$$

of the group C, satisfies the resolvent

$$R_C: \qquad \left(\frac{-w'}{2w}\right)' + \left(\frac{-w'}{2w}\right)^2 + I = 0.$$

If A is the (rationality) group of the equation $K(y) = 0$ in a domain $D\{x\}$, then by the Picard-Vessiot Double Theorem, Δ, the invariant of the group A, must have a rational value. Therefore the resolvent R_A has a rational integral; and inversely.

Similarly, if $K(y)$ has for its rationality group either B, C, D, E, F, (G) or H, the resolvent R_B has a rational integral; and inversely. Indeed, if the group of the equation is D or one of its subgroups, R_B has *two* rational integrals,

$$u_1 = \frac{y_1'}{y_1} + \frac{1}{2}k_1, \qquad u_2 = \frac{y_2'}{y_2} + \frac{1}{2}k_2;$$

and inversely.

If $K(y)$ has C for its rationality group, then R_C has a rational integral; and if it has F for its group, then R_F has a rational integral; and inversely.

Our procedure will be: (1) to set up the resolvents for each equation I, II, III, IV, V, VI; (2) to determine which have or have not rational integrals (for the domains listed in §1), and thus: (3) the rationality groups of the equations for these domains.

The notation that will be used is an obvious one, R_A^I denoting the resolvent R_A with the value of I taken from equation I, R_B^{III} the resolvent R_B with the value of I taken from the equation III, and so forth.

5. Equation with Constant coefficients. For the equation I we have, $I = a_2 - \frac{a_1^2}{4}$, and the resolvents become

$$R'_A: \quad \Delta' + a_1 \Delta = 0,$$

$$R'_B: \quad u' + u^2 + \left(a_2 - \frac{a_1^2}{4}\right) = 0,$$

$$R'_C: \quad \left(\frac{-w'}{2w}\right)' + \left(\frac{-w'}{2w}\right)^2 + \left(a_2 - \frac{a_1^2}{4}\right) = 0,$$

$$R'_F: \quad \left(\frac{-v''}{2v'}\right)' + \left(\frac{-v''}{2v'}\right)^2 + \left(a_2 - \frac{a_1^2}{4}\right) = 0.$$

(a) *Let the domain of rationality be R, the totality of real numbers.*

Case 1. $a_1 \neq 0$, $I > 0$. Since $e^{-a_1 x}$ is irrational (not in the domain R), the resolvent R'_A has an irrational integral $\Delta = e^{-a_1 x}$; this excludes the groups A and G. We have for integrals of R'_A

$$u_1 = i\sqrt{I}, \quad u_2 = -i\sqrt{I},^* \quad u_3 = -\sqrt{I}\tan x\sqrt{I}, \ (i = \sqrt{-1}),$$

none of which is rational. Therefore the rationality group of the equation I cannot be B, C, D, E, F, (G), H, and must be the general linear homogeneous group L_{22}.

Illustrative equation:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0.$$

Case 2. $a_1 = 0$, $I > 0$. In this case R'_A has a rational integral, $\Delta = 1$, and the group of the equation is the special linear group A.

Illustrative equation:

$$\frac{d^2y}{dx^2} + 5y = 0.$$

Case 3. $a_1 \neq 0$, $I = 0$. The resolvent R'_B has a rational integral $u_1 = 0$, and the resolvant R'_C has a rational integral $w = 1$. Therefore the rationality group of the equation is the group C.

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* Unlike the case in linear equations, u_1 and u_2 must be regarded as distinct; and indeed, they lead to different integrals of I,

$$y_1 = e^{\frac{-a_1 x}{2}} e^{u_1 x}, \quad y_2 = e^{\frac{-a_1 x}{2}} e^{u_2 x}.$$

Illustrative equation:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0.^*$$

Case 4. $a_1 = 0$, $I = 0$. The group of the equation in the domain R is G.

Illustrative equation:

$$\frac{d^2y}{dx^2} = 0.$$

Case 5. $a_1 \neq 0$, $I < 0$. The resolvent R_H^I has two rational integrals,

$$u_1 = \sqrt{-I}, \qquad u_2 = -\sqrt{-I}.$$

The group of the equation must be D, F, or H. One can exclude F by noting that R_F^I does not have an integral in the domain R. We could set up the resolvent for the invariant $\frac{y_2^a}{y_1^b}$ of H and find under what conditions it has a rational integral. I have not completed this calculation on account of its extreme length; however, the result may be obtained in the following simple (though irregular) manner. The integrals of I are

$$y_1 = e^{\left(-\frac{a_1}{2} + \sqrt{-I}\right)x}, \qquad y_2 = e^{\left(-\frac{a_1}{2} - \sqrt{-I}\right)x},$$

and it can be seen that

$$\frac{y_2^a}{y_1^b} = 1$$

when $a = k(a_1 - 2\sqrt{-I})$, $b = k(a_1 + 2\sqrt{-I})$, k denoting any number in the domain R. Therefore, if $k(a_1 - 2\sqrt{-I})$ and $k(a_1 + 2\sqrt{-I})$ are integers, the group of the equation is H, while if $k(a_1 - 2\sqrt{-I})$ and $k(a_1 + 2\sqrt{-I})$ are not integers, the group of the equation is D.

Illustrative equation:

$$\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 9y = 0, \text{ group } H \text{ with } a = 1, \quad b = 9.$$

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* Two integrals are of course, $y_1 = e^{4x}$, $y = xe^{4x}$; and it is interesting to actually verify that the invariants of the other group (§3) are irrational; thus an invariant of F, $y_2/y_1 = x$, is not in the domain R.

Illustrative equation :

$$\frac{d^2y}{dx^2} + \pi \frac{dy}{dx} + y = 0, \quad \text{group } D.$$

Case 6. $a_1 = 0$, $I < 0$. The resolvent R_A^I has the rational integral $\Delta = 1$, and the resolvent R_B^I has two rational integrals

$$u_1 = \sqrt{-I}, \quad u_2 = -\sqrt{-I}.$$

The greatest subgroup common to A and D, viz :

H': $$y_1p_1 - y_2p_2$$

is the group of the equation. H' is the group H, with $a = -b = 1$.

Illustrative equation : $\frac{d^2y}{dx^2} - 4y = 0$.

b). *Let the domain of rationality be C, the totality of complex numbers.* In this domain we do not need to distinguish between $I < 0$ and $I > 0$, since in either event, $\sqrt{\pm I}$ is rational (lies in the domain C).

Case 1. $a_1 \neq 0$, $I = a_2 - a_1^2/4 \neq 0$. The resolvent R_B^I has two rational integrals

$$u_1 = \sqrt{-I}, \quad u_2 = -\sqrt{-I}.$$

As in a), case 5, the group of the equation is H if $k(a_1 - 2\sqrt{-I})$ and $k(a_1 + 2\sqrt{-I})$ are integers, and it is the group D if these numbers are not integers. The illustrative equations of a) case 5 can serve here also.

Case 2. $a_1 = 0$, $I \neq 0$. As in a), case 6, the group of the equation is $H' \equiv y_1p_1 - y_2p_2$. Same illustrative equation as in a), case 6.

Case 3. $a_1 \neq 0$, $I = 0$. The resolvents R_B^I and R_C^I have each one rational integral, $u_1 = 0$, $w_1 = 1$; therefore the group of the equation is C.

Illustrative equation :

$$\frac{d^2y}{dx^2} + 2(1+i)\frac{dy}{dx} + 2iy = 0.$$

Case 4. $a_1 = 0$, $I = 0$. Same as a), case 4.

c) *Let the domain of rationality be $R[x]$, the totality of rational functions of x with real coefficients.*

Case 1. $a_1 \neq 0$, $I > 0$. Same as a), case 1.

Case 2. $a_1 = 0$, $I > 0$. Same as a), case 2.

Case 3. $a_1 \neq 0$, $I = 0$. The resolvent R'_B has two rational integrals, $u_1 = 0$, $u_2 = 1/x$ in the domain $R[x]$, and the resolvent R'_F has a rational integral $v = x$; therefore the group of the equation is F.*

Case 4. $a_1 = 0$, $I = 0$. The resolvent R'_A has a rational integral $\Delta = 1$, the resolvent R'_B has two rational integrals $u_1 = 0$, $u_2 = 1/x$ and R'_F has a rational integral $v = x$; there is no subgroup common to A and F except the identical transformation. Therefore the group of the equation is the *Identity*.

Case 5. $a_1 \neq 0$, $I < 0$. Same as a), case 5.

Case 6. $a_1 = 0$, $I < 0$. Same as a), case 6.

d) *Let the domain of rationality be $C[x]$, the totality of rational functions of x with real or complex coefficients.*

Case 1. $a_1 \neq 0$, $I \neq 0$. Same as b), case 1.

Case 2. $a_1 = 0$, $I \neq 0$. Same as b), case 2.

Case 3. $a_1 \neq 0$, $I = 0$. Same as c), case 3.

Case 4. $a_1 = 0$, $I = 0$. Same as c), case 4.

e) *Let the domain of rationality be $R[x, e^x]$.* In this domain the resolvent R'_A has the rational integral $\Delta = e^{-a_1 x}$, and the rationality group of the equation must be A or one of the subgroups.

Case 1. $I > 0$. Since R'_B does not have a rational integral, the group of the equation is A.
Illustrative equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 0.$$

Case 2. $I = 0$. The resolvent R'_B has two rational integrals, $u_1 = 0$, $u_2 = 1/x$ and the resolvent R'_F has two rational integrals $v_1 = x$, $v_2 = -1/x$. The only subgroup common to A and F is the *Identity*, which must be the group of the equation.

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* The resolvent R'_C has a rational integral $w = 1$, but this is also a consequence of the fact that $w = dv/dx$.

Illustrative equation:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0.$$

Case 3. $I < 0$. The resolvent R_A^I has one rational integral $\Delta = e^{-a_1 x}$;

R_B^I has two rational integrals $v_1 = \frac{1}{2\sqrt{-I}} e^{2x\sqrt{-I}}$, $v_2 = \frac{1}{2\sqrt{-I}} e^{-2x\sqrt{-I}}$.

The group of the equation is the *Identity*.

f) *Let the domain of rationality be* $C[x, e^x] \equiv C[x, \sin x]$. In this domain the group of the equation with constant coefficients is the *Identity*. In domains larger than $C[x, e^x]$, as for instance $C[x, e^x, f(x)]$ the group of the equation I is always *Identity*.

In smaller domains, such as $C[e^x]$, $R[e^x]$, the domain of rational numbers, etc., certain slight modifications of the above investigation are needed, and are easily made.

6. The Hypergeometric Equation. For the equation II we have

$$I_i = \frac{h_1}{x} + \frac{h_2}{x^2} + \frac{k_1}{1-x} + \frac{k_2}{(1-x)^2},$$

where

$$h_1 = k_1 = -\alpha\beta - \tfrac{1}{2}\gamma(\gamma - \alpha - \beta - 1),$$

$$h_2 = -\left(\frac{\gamma}{2}\right)^2 + \frac{\gamma}{2},$$

$$k_2 = -\left(\frac{\gamma - \alpha - \beta - 1}{2}\right)^2 - \frac{\gamma - \alpha - \beta - 1}{2}.$$

The principal resolvents from which the rationality groups will be deduced are:

$$R_A^{II}: \quad \frac{d\Delta}{dx} + \frac{\gamma - (\alpha + \beta + 1)x}{x(1-x)}\Delta = 0,$$

$$R_B^{II}: \quad \frac{du}{dx} + u^2 + \frac{h_1}{x} + \frac{h_2}{x^2} + \frac{k_1}{1-x} + \frac{k_2}{(1-x)^2} = 0,$$

$$R_C^{II}: \quad \left(\frac{-w'}{2w}\right)' + \left(\frac{-w'}{2w}\right)^2 + \frac{h_1}{x} + \frac{h_2}{x^2} + \frac{k_1}{1-x} + \frac{k_2}{(1-x)^2} = 0,$$

$$R_F^{II}: \quad \left(\frac{-v''}{2v'}\right)' + \left(\frac{-v''}{2v'}\right)^2 + \frac{h_1}{x} + \frac{h_2}{x^2} + \frac{k_1}{1-x} + \frac{k_2}{(1-x)^2} = 0.$$

The relations between the solutions of R''_B, R''_C, R''_F are useful:

$$w = e^{-2\int u dx}, \qquad v = \int w dx.$$

We must find under what conditions the resolvents have rational integrals.

a) R''_A will have its integral

$$\Delta - \frac{(1-x)^{\gamma-(\alpha+\beta+1)}}{x^\gamma}$$

rational, if the exponents $\gamma - (\alpha + \beta + 1)$ and γ are integers; hence it will be rational if

$$\alpha + \beta = \text{integer, and,} \quad \gamma = \text{integer.}$$

These integers may be positive or negative.

b) It is known* that R''_B, has a rational integral

$$u = \frac{a_1}{x} \frac{a_2}{1-x} + \frac{g'(x)}{g(x)},$$

provided that one of the four quantities

$$\alpha, \beta, \gamma - \alpha, \gamma - \beta$$

is a negative integer. Here a_1 and a_2 have the values

$$a_1' = \frac{\gamma}{2}, \qquad a_1'' = 1 - \frac{\gamma}{2},$$

$$a_2' = \frac{\gamma - \alpha - \beta - 1}{2}, \qquad a_2'' = -1 - \frac{\gamma - \alpha - \beta - 1}{2},$$

and $g(x)$ is a polynomial (without multiple roots) of degree n, where n has one of the values

$$-\alpha, -\beta; \ -\alpha + 1, \ -\beta + 1; \ \alpha - \gamma; \ \beta - \gamma; \ \alpha - \gamma + 1, \ \beta - \gamma + 1.$$

c) Since $w = e^{-2\int u dx}$, we have

$$w = \frac{(1-x)^{2a_2}}{x^{2a_1} g^2},$$

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* E. Beke, Die Irreducibilität der homogenen linearen Differentialgleichungen, *Mathematische Annalen*, vol. 45 (1894), p. 287. See also Picard's *Traité d'Analyse*, 2nd edition, vol. 3, p 561.

Since the part of Beke's paper (§6) which bears directly on the present investigation is valid in the domains $R[x]$ and $C[x]$, the results here deduced are valid in the same domains

and the integral w of R''_C will be rational if u is a rational function and if $2a_1$ and $2a_2$ are integers, that is, if

$$u = \text{rat.}\, f(x) \quad \text{and} \quad \gamma = \text{integer}, \quad \alpha + \beta = \text{integer}.$$

By comparing with (a) we are led to an interesting conclusion: if R''_A and R''_B have each a rational integral, then R''_C has also a rational integral. The group common to A, B and C being G, we see that: *if R''_A and R''_B have each a rational integral, the group of the equation will be G.*

f) To determine whether R''_F has a rational integral, we consider that

$$v = \int w dx,$$

and

$$w = \frac{(1-x)^{2a_2}}{x^{2a_1} g^2},$$

where a_1 and a_2 are known integers and g a known polynomial. Therefore, for any particular problem, the determination of w is a straightforward process which can be completed in a finite number of steps.

Let the domain of rationality be either $R[x]$ or $C[x]$.

Case 1. The rationality group will be the general linear homogeneous group L_{2^2} if each of the following conditions is unfulfilled:

z) $\alpha + \beta = \text{integer}$, $\gamma = \text{integer}$;

y) $\alpha = \text{negative integer}$;

x) $\beta = \text{negative integer}$;

w) $\gamma - \alpha = \text{negative integer}$;

v) $\gamma - \beta = \text{negative integer}$.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{6-9x}{2x(1-x)} - \frac{3}{x(1-x)}\, y = 0.$$

Case 2. The rationality group of the equation will be the special linear group if

z) $\alpha + \beta = \text{integer}$, $\gamma = \text{integer}$,

and if each of the other conditions of case 1, (y, x, w, v) is unfulfilled.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{1-5x}{x(1-x)} - \frac{15}{4x(1-x)}\, y = 0.$$

Case 3. If α is a negative integer, and if every other condition of case 1, (z, x, w, v) is unfulfilled, R''_B will have exactly one rational integral, and the group of the equation will be B.

Similar remarks apply to the possibilities shown in the following table; each of the equations has B for its group.

NEGATIVE INTEGER	UNFULFILLED	ILLUSTRATIVE EQUATION
α	z, x, w, v	$y'' + \frac{4+x}{2x(1-x)}\, y' + \frac{10y}{x(1-x)} = 0$
β	z, y, w, v	$y'' + \frac{4+x}{2x(1-x)}\, y' + \frac{10y}{x(1-x)} = 0$
$\gamma - \alpha$	z, y, x, v	$y'' + \frac{11x-8}{2x(1-x)}\, y' - \frac{5y}{x(1-x)} = 0$
$\gamma - \beta$	z, y, x, v	$y'' + \frac{11x-8}{2x(1-x)}\, y' - \frac{5y}{x(1-x)} = 0$

The first and second illustrations of this table are purposely taken alike, and similarly for the third and fourth. In the first and second we have

$$\gamma = 2, \quad \alpha + \beta + 1 = -\tfrac{1}{2}, \quad \alpha\beta = -10,$$

whence,

$$\alpha = -4, \ \beta = \tfrac{5}{2}, \qquad \gamma = 2,$$

the first illustration; or

$$\alpha = \tfrac{5}{2}, \quad \beta = -4, \quad \gamma = 2,$$

the second illustration.

Case 4. Let α = negative integer, β = negative integer (different from α), and $\gamma \neq$ integer. R''_B has now two rational integrals, and the group of the equation is D. The same remarks apply to the possibilities shown in the following table:

NEG. INTEGERS (DIFFERENT)	NOT INTEGER	ILLUSTRATIVE EQUATION
α, β	γ	$y'' + \frac{5+8x}{2x(1-x)}y' - \frac{6}{x(1-x)}y = 0$
α, $\gamma-\alpha$	β	$y'' - \frac{6+5x}{2x(1-x)}y' + \frac{5}{2x(1-x)}y = 0$
α, $\gamma-\beta$	β	$y'' + \frac{3-5x}{2x(1-x)}y' + \frac{5}{2x(1-x)}y = 0$
β, $\gamma-\alpha$	α	$y'' + \frac{3-5x}{2x(1-x)}y' + \frac{5}{2x(1-x)}y = 0$
β, $\gamma-\beta$	α	$y'' - \frac{6+5x}{2x(1-x)}y' + \frac{5}{2x(1-x)}y = 0$
$\gamma-\alpha$, $\gamma-\beta$	γ, and $\alpha \neq$ neg. integer $\beta \neq$ neg. integer	$y'' + \frac{1-10x}{2x(1-x)}y' - \frac{15}{4x(1-x)}y = 0$

We shall not stop here to enumerate the large additional number of special cases that may arise. By following out the method of this section, it is possible to determine the group of any other given hypergeometric equation in the domains $R[x]$ and $C[x]$. We give two important instances in §7 and §8.

In the event of R_B'' having two rational integrals a special investigation is needed. Ordinarily, the group of the equation will be D, but if, in addition, $v = y_2/y_1$ is rational, the group of the equation is F. However, it may happen that the group of the equation is H. Let $t = y_2^a/y_1^b$ be the third invariant of H, then

$$\frac{t'}{t} = a\frac{y_2'}{y_2} - b\frac{y_1'}{y_1} = au_2 - bu_1 + \frac{ak_1}{2} - \frac{bk_1}{2},$$

and

$$t = e^{\int\left(au_2 - bu_1 + \frac{ak_1}{2} - \frac{bk_1}{2}\right)dx};$$

if now, this value lies in the domain of rationality, the group of the equation is H.

7. The Equation for the Moduli of Periodicity of the Elliptic Integrals of the First Kind. The equation III is a hypergeometric equation for which

$$\alpha = \beta = \tfrac{1}{2}, \quad \gamma = 1.$$

The only condition of §6, case 1, which is fulfilled is z. Therefore the resolvent R_A^{III} has a rational integral

$$\Delta = \frac{1}{x(1-x)},$$

and none of the other resolvents have rational integrals (indeed, the fact R_B^{II} does not have a rational integral is decisive, for it excludes groups B, C, D, E, F, G, H). The rationality group of the equation III is A in the domains $R[x]$ and $C[x]$.

8. Legendre's Polynomial of order k. Number IV is a hypergeometric equation for which

$$a = k + 1, \quad \beta = -k, \quad \gamma = 1.$$

Here we have fulfilled the conditions

z) $a + \beta =$ integer, $\gamma =$ integer,

x) $\beta =$ negative integer,

w) $\gamma - a =$ negative integer.

(z) shows that R_A^{IV} has a rational integral, while (x) and (w) show that R_B^{IV} has two rational integrals. It follows (§6, c) that R_C^{IV} has two rational integrals. The rationality group of the equation is therefore the sub-group common to A, D, C, that is, the *Identity*.

9. Bessel's Equation. For the equation V we have

$$I = 1 + \frac{1}{x^2}\left(\frac{1}{4} - n^2\right),$$

and the corresponding resolvents are

$$R_A^V: \quad \Delta' + \frac{1}{x}\Delta = 0,$$

$$R_B^V: \quad u' + u^2 + 1 + \frac{1}{x^2}\left(\frac{1}{4} - n^2\right) = 0,$$

$$R_C^V: \quad \left(\frac{-w'}{2w}\right)' + \left(\frac{-w'}{2w}\right)^2 + 1 + \frac{1}{x^2}\left(\frac{1}{4} - n^2\right) = 0.$$

$$R_F^V: \quad \left(\frac{-v''}{2v'}\right)' + \left(\frac{-v''}{2v'}\right)^2 + 1 + \frac{1}{x^2}\left(\frac{1}{4} - n^2\right) = 0.$$

The Integral of R_A^v. The resolvent R_A^v has for an integral

$$\Delta = \frac{1}{x}.$$

The Integrals of R_B^v. A solution of the resolvent R_B^v has $x = 0$ for a pole. If it has another pole, $x = a$, one can see from the form of the resolvent that the coefficient of $(x - a)^{-1}$ in u, must be unity. Therefore the solution must be of the form

$$u_1 = i + \frac{h_1}{x} + \frac{g'(x)}{g(x)} \quad \text{or} \quad u_2 = -i + \frac{h_2}{x} + \frac{g'(x)}{g(x)},$$

where the g's are polynomials.

By substituting in R_B^v we find, first of all, that h_1 and h_2 can have either of two values

$$h_1' = h_2' = \tfrac{1}{2} + n,$$
$$h_1'' = h_2'' = \tfrac{1}{2} - n.$$

The substitution of u_1 and u_2 in R_B^v gives us, for the determination of g

$$xg'' + (2ix + 1 + 2n)\, g' + i(1 + 2n)\, g = 0,$$
$$xg'' + (2ix + 1 - 2n)\, g' + i\,(1 - 2n)\, g = 0,$$
$$xg'' + (-\, 2ix + 1 + 2n)\, g' - i(1 + 2n)\, g = 0,$$
$$xg'' + (-\, 2ix + 1 - 2n)\, g' - i(1 - 2n)\, g = 0.$$

If p is the degree of g, the highest power of x in these equations gives us

$$2ip + i(1 + 2n) = 0,$$
$$2ip + i(1 - 2n) = 0,$$
$$-\, 2ip - i(1 + 2n) = 0,$$
$$-\, 2ip - i(1 - 2n) = 0.$$

If g exists at all, p is an integer, and therefore $2n$ is an odd integer. The other powers of x in the equations for g, equated to zero, will give us the coefficients of the polynomials, step by step. In this manner we reach the conclusion that *the resolvent R_B^v has two rational integrals*

$$u_1 = i + \frac{\frac{1}{2} + n}{x} + \frac{g_1'(x)}{g_1(x)}, \qquad u_2 = -i + \frac{\frac{1}{2} + n}{x} + \frac{g_2'(x)}{g_2(x)};$$

or

$$u_1 = i + \frac{\frac{1}{2} - n}{x} + \frac{g_3'(x)}{g_3(x)}, \qquad u_2 = -i + \frac{\frac{1}{2} - n}{x} + \frac{g_4'(x)}{g_4(x)},$$

where the g's are determinable polynomials, if $2n$ is an odd integer.

The Integrals of R_C^v. Since $w = e^{-2\int u dx}$ (§4), u denoting a solution of R_B^v, we have

$$w_1 = e^{-2ix}\frac{x^{-1 \pm 2n}}{g^2} = \frac{(\cos 2x - i \sin 2x)x^{-1 \pm 2n}}{g^2},$$

$$w_2 = \frac{(\cos 2x + i \sin 2x)x^{-1 \pm 2n}}{g^2},$$

provided that g exists; that is, provided $2n$ is an odd integer. In particular, if $2n = \pm 1$, it is easily verified that

$$w_1 = \cos 2x - i \sin 2x,$$

$$w_2 = \cos 2x + i \sin 2x$$

are solutions of R_B^v.

The integrals of R_F^v. We have (§4) $v = \int w dx$. Therefore, if $2n$ is an odd integer (i.e. if g exists), resolve $\frac{x^{-1 \pm 2n}}{g^2}$ into partial fractions. Because $\int \frac{\cos 2x\, dx}{x - a}$, $\int \frac{\cos 2x\, dx}{x - a}$ cannot be expressed rationally in terms of x and $\cos x$, the resolvent R_F^v does not have an integral in the domains discussed in this section, $R[x]$, $C[x]$, $R[x, \sin x]$, $C[x, \sin x]$.

However, if $2n = \pm 1$,

$$v_1 = \cos 2x - i \sin 2x,$$

$$v_2 = \cos 2x + i \sin 2x,$$

are solutions of R_F^v.

a) *Let the domain of rationality be $R[x]$.* The resolvent R_A^v is the only one which has a rational integral and therefore the group of the equation is A.

Illustrative equation:

$$y'' + \frac{y'}{x} + \left(1 - \frac{n^2}{x^2}\right)y = 0, \qquad n = \text{any real number}.$$

b) *Let the domain of rationality be* $C[x]$.

Case 1. $2n \neq$ odd integer. The only one of the resolvents which has a rational integral is R_A^v. Therefore the group of the equation is A.

Illustrative equation:

$$y'' + \frac{y'}{x} + \left(1 - \frac{4}{x^2}\right)y = 0.$$

Case 2. $2n =$ odd integer. The resolvent R_A^v has a rational integral and the required group is a sub-group of A.

The resolvent R_B^v has two rational integrals and (by §§3 and 4) the required group is a sub-group of D. Therefore the required group is the greatest sub-group common to A and D, i. e., the group of the equation is

$$y_1 p_1 - y_2 p_2.$$

This is the group H with $a = -b = 1$.

c) *Let the domain of rationality be* $R[x, \sin x]$. The same discussion and result as in a) applies.

d) *Let the domain of rationality be* $C[x, \sin x]$.

Case 1. $2n \neq$ odd integer. In this case R_A^v is the only resolvent having a rational integral, and the group of the equation is A.

Illustrative equation:

$$y'' + \frac{y'}{x} + \left(1 - \frac{4}{x^2}\right)y = 0.$$

Case 2. $2n =$ odd integer.

R_A^v has one rational integral, giving the group A;

R_B^v has two rational integrals, giving the group D;

R_C^v has two rational integrals, giving the group C.

The greatest sub-group common to A, C, D is the group of the equation. This group is the *Identity*.

Illustrative equation:

$$y'' + \frac{y'}{x} + \left(1 - \frac{9}{x^2}\right)y = 0.$$

10. Cauchy's Equation. For the equation VI we have

$$I = \frac{4a_2 - a_1^2 + 2a_1}{4x^2},$$

and the corresponding resolvents are

$$R_A^{VI}: \quad \Delta' + \frac{a_1}{x}\Delta = 0,$$

$$R_B^{VI}: \quad u' + u^2 + \frac{4a_2 - a_1^2 + 2a_1}{4x^2} = 0,$$

$$R_C^{VI}: \quad \left(\frac{-w'}{2w}\right)' + \left(\frac{-w'}{2w}\right)^2 + \frac{4a_2 - a_1^2 + 2a_1}{4x^2} = 0,$$

$$R_F^{VI}: \quad \left(\frac{-v''}{2v'}\right)' + \left(\frac{-v''}{2v'}\right)^2 + \frac{4a_2 - a_1^2 + 2a_1}{4x^2} = 0.$$

An integral of R_A^{VI} is

$$\Delta = \frac{1}{x^{a_1}}.$$

Two integrals of R_B^{VI} are

$$u_1 = \frac{P_1}{x}, \quad u_2 = \frac{P_2}{x}$$

where

$$P_1 = \frac{1 + \sqrt{(a_1 - 1)^2 - 4a_2}}{2}, \quad P_2 = \frac{1 - \sqrt{(a_1 - 1)^2 - 4a_2}}{2}.$$

Two integrals of R_C^{VI} are

$$w_1 = \frac{1}{x^{2P_1}}, \quad w_2 = \frac{1}{x^{2P_2}}.$$

Two integrals of R_F^{VI} are

$$v_1 = \frac{x^{1-2P_1}}{1 - 2P_1}, \quad v_2 = \frac{x^{1-2P_2}}{1 - 2P_2}.*$$

From these integrals we determine the group of Cauchy's equation for each case that arises.

* When $P_1 = P_2 = \frac{1}{2}$, then $v_1 = v_2 = \log x$.

a–b) *Let the domain of rationality be R or C.*

Case. 1 $a_1 \neq 0$, $P_1 \neq 0$, $P_2 \neq 0$. None of the resolvents have a rational integral and the group of the equation is L_{22}.

Illustrative equation :

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \frac{1}{x^2}\, y = 0.$$

Case 2. $a_1 = 0$, $P_1 \neq 0$, $P_2 \neq 0$. R_A^{VI} has a rational integral $\Delta = 1$ and the group of the equation is A.

Illustrative equation :

$$\frac{d^2y}{dx^2} + \frac{1}{x^2}\, y = 0.$$

Case 3. $a_1 \neq 0$, $P_1 \neq 0$, $P_2 = 0$. R_B^{VI} has a rational integral $u_2 = 0$ and R_C^{VI} has a rational integral $u_2 = 1$. Therefore the group of the equation is C.

Illustrative equation :

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{4x^2}\, y = 0.$$

It should be noted that if $P_2 = 0$, then $P_1 = 1$. Also, case 3 is equivalent to the conditions

$$a_1 \neq 0, \quad P_1 = 0, \quad P_2 \neq 0.$$

Case 4. $a_1 = 0$, $P_1 \neq 0$, $P_2 = 0$. R_A^{VI}, R_B^{VI} and R_C^{VI} have rational integrals. The sub-group common to A and C, viz: G, is the group of the equation.

Illustrative equation : $\dfrac{d^2y}{dx^2} = 0.$

Case 4 is equivalent to the conditions $a_1 = 0$, $P_1 = 0$, $P_2 \neq 0$.

It is impossible for P_1 and P_2 to vanish simultaneously so that no further cases can arise.

c) *Let the domain of rationality be $R[x]$.*

Case 1. $a_1 \neq$ integer, $(a_1 - 1)^2 - 4a_2 < 0$. None of the resolvents have rational integrals and the group of the equation is L_{22}.

Illustrative equation :

$$\frac{d^2y}{dx^2} + \frac{1}{2x}\frac{dy}{dx} + \frac{1}{x^2}\, y = 0.$$

Case 2. $a_1 =$ integer, $(a_1 - 1)^2 - 4a_2 < 0$. The group of the equation is A.
Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{2}{x^2}y = 0.$$

Case 3. $a_1 \neq$ integer, $(a_1 - 1)^2 - 4a_2 = 0$. Here R_B^{vI} has a rational integral $u = \frac{1}{2x}$; R_C^{vI} has a rational integral $w = \frac{1}{x}$. The group of the equation is C.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{5}{2x}\frac{dy}{dx} + \frac{9}{16x^2}y = 0.$$

Case 4. $a_1 =$ integer, $(a_1 - 1)^2 - 4a_2 = 0$. The subgroup common to A and C, viz: G, is the group of the equation. We have to consider two subcases:

$4_1)$ $a_1 =$ odd integer. $u = \frac{y_1'}{y_1} + \frac{a_1}{2x} = \frac{1}{2x}$, whence $y_1 = x^{\frac{1-a_1}{2}}$, and since $\frac{1 - a_1}{2} =$ integer, the finite equations of G,

$$G_1: \ \eta_1 = y_1, \quad \eta_2 = ty_1 + y_2$$

give the rationality group of the equation.
Illustrative equation

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{1}{x^2}y = 0.$$

$4_2)$ $a_1 =$ even integer. Since $\frac{1 - a_1}{2} = \frac{\text{odd integer}}{2}$, we have $y_1 = \sqrt{x^{\text{odd power}}}$.

In this case the rationality group is *complex*,* and its finite substitutions are

$$G_2: \quad \begin{array}{ll} \eta_1 = y_1, & \eta_1 = y_1 \\ \eta_2 = ty_1 + y_2; & \eta_2 = -ty_2 + y_2. \end{array}$$

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{4}{x}\frac{dy}{dx} + \frac{9}{4x^2}y = 0.$$

Case 5. $u_1 \neq$ integer, $(a_1 - 1)^2 - 4a_2 > 0$. Here u_1 and u_2 are both rational and the group of the equation is D or one of its subgroups. We have to consider two subcases.

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* Vessiot, loc. cit., p. 261, Remark I; also p. 271.

5_1). $2P_1$ and $2P_2 \neq$ integers. Since the integrals of R_C^{VI} and R_F^{VI} are both irrational, the group of the equation is D.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{5}{4x}\frac{dy}{dx} - \frac{1}{8x^2}y = 0.$$

5_2). $2P_1$ and $2P_2 =$ integers $\neq 1$*. Since R_C^{VI} and R_F^{VI} have rational integrals, the group of the equation is F.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{5}{2x}\frac{dy}{dx} - \frac{91}{16x^2}y = 0.$$

It cannot happen that $2P_1 =$ integer, $2P_2 \neq$ integer; so this case need not be discussed.

Case 6. $a_1 =$ integer, $(a_1 - 1)^2 - 4a_2 > 0$.

6_1) $2P_1$ and $2P_2 \neq$ integers. This is 5_1) with R_A^{VI} having a rational integral. The subgroup common to D and A, viz:

H') $$y_1p_1 - y_2p_2$$

is the group of the equation.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{5}{x}\frac{dy}{dx} + \frac{2}{x^2}y = 0.$$

6_2). $2P_1$ and $2P_2 =$ integers. The group of the equation is the *Identity*.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{5}{x}\frac{dy}{dx} + \frac{3}{x^2}y = 0.$$

d) *Let the domain of rationality be* $C[x]$. In this domain $\sqrt{(a_1 - 1)^2 - 4a_2}$ is rational whether the quantity under the radical is positive or negative. Therefore c), case 1, becomes identical with c), case 5; and c), case 2, becomes identical with c), case 6. The discussion for the domain $C[x]$ is thus the same as the discussion for the domain $R[x]$, cases 3, 4, 5, 6.

e) *Let the domain of rationality be* $C[x, \log x]$.

Case 1. $a_1 \neq$ integer, $(a_1 - 1)^2 - 4a_2 \neq 0$. Same discussion as c), case 5.

Case 2. $a_1 =$ integer, $(a_1 - 1)^2 - 4a_2 \neq 0$. Same discussion as c), case 6.

* $2P_1 = 1$ means $(a_1 - 1)^2 - 4a_2 = 0$; a case already discussed.

Case 3. $a_1 \neq$ integer, $(a_1 - 1)^2 - 4a_2 = 0$. The resolvent R_B^{VI} has one rational integral $u = 1/(2x)$; the resolvent R_C^{VI} has one rational integral $w_1 = 1/x$, and the resolvent R_F^{VI} has one rational integral $v_1 = \log x$. The sub-group common to B, C, F, that is, F, is the group of the equation.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{5}{2x}\frac{dy}{dx} + \frac{9}{16x^2}y = 0.$$

Case 4. $a_1 =$ integer, $(a_1 - 1)^2 - 4a_2 = 0$. The group of the equation being the sub-group common to A and F has no infinitesimal transformation.

4_1. If $a_1 =$ odd integer, the group of the equation is the *Identity*.

Same illustrative equation as c) case 4_1.

4_2). If $a_1 =$ even integer, the group of the equation is a finite one, composed of the two substitutions: Identity and $\eta_1 = -y_1$, $\eta_2 = y_2$.

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ENVELOPES OF ONE-PARAMETER FAMILIES OF PLANE CURVES

By W. J. Risley and W. E. MacDonald

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1. *Definition of an Envelope.* p. 76.

Let a one-parameter family of plane curves be defined by $y = f(x, a)$, where $f(x, a)$ is analytic in x and a, and $f_a(x, a) \not\equiv 0$ in the neighborhood of (x_0, a_0). If there exists a curve E, defined by $x = \phi(a)$, $y = \psi(a)$, where $\phi(a)$ and $\psi(a)$ are analytic and $|\phi'| + |\psi'| > 0$ in the neighborhood of a_0, and if, for an arbitrary value a' of a in the neighborhood of a_0, the curve $y = f(x, a')$ is tangent to E at a', then E is an envelope of the curves of the family.

2. *Necessary Conditions.* p. 76.

The functions $x = \phi(a)$, $y = \psi(a)$ must satisfy the equations $y - f(x, a) = 0$ and $f_a(x, a) = 0$.

3. *Sufficient Conditions for a Smooth Envelope.* p. 77.

If, in the region $|x - x_0| < h$, $|a - a_0| < k$, 1) $f(x, a)$ is analytic; 2) $f_a(x, a) \not\equiv 0$; 3) $f_a(x_0, a_0) = 0$; 4) $f_{a,x}(x_0, a_0) \neq 0$; 5) $f_{a,a}(x_0, a_0) \neq 0$; then the family of curves, $y = f(x, a)$, has an envelope E, determined by the equations a) $f_a(x, a) = 0$, b) $y - f(x, a) = 0$.

4. The preceding conditions too restrictive. p. 79.

5. *Weierstrass's Implicit Function Theorem.* p. 80.

Let $F(x, y)$ be analytic in x and y at (x_0, y_0), $F(x_0, y) \not\equiv 0$, and $F(x_0, y_0) = 0$; then $F(x, y)$ may be written $F(x, y) \equiv P(x, y) \cdot H(x, y)$, where $P(x, y) \equiv (y - y_0)^n + A_1(x)(y - y_0)^{n-1} + \ldots + A_n(x)$, where $A_i(x)$ is analytic at x_0, where $A_i(x_0) = 0$, where $H(x, y)$ is analytic at (x_0, y_0) and where $H(x_0, y_0) \neq 0$.

Reducibility Theorem. p. 81.

A necessary and sufficient condition that $P(x, y) \equiv y^n + A_1(x)\, y^{n-1} + \ldots + A_n(x)$, where $A_i(x)$ is analytic at x_0, and where $A_i(x_0) = 0$, be irreducible at (x_0, y_0) is that, no matter how small a neighborhood of x_0 one consider, the Riemann surface for the function y of x, defined by $P(x, y) = 0$, spread out over this neighborhood, consist of a single piece.

6. *Applications of the last two Theorems.* p 82.

Fundamental Theorem. p. 86.

Given a one-parameter family of curves $y = f(x, a)$ where $f(x, a)$ is a function, analytic at (x_0, a_0), such that

1) $\qquad f_{a, x^i}(x_0, a) \equiv 0, \qquad i = 0, 1, 2, \ldots, m-1,$

but $\qquad f_{a, x^m}(x_0, a) \not\equiv 0,$

2) $\qquad f_{a^j}(x, a_0) \equiv 0, \qquad j = 1, 2, 3, \ldots, n,$

but $\qquad f_{a^{n+1}}(x, a_0) \not\equiv 0;$

then if

3) $\qquad f_{a^{n+1}, x^m}(x_0, a_0) \neq 0,$

the curves $y = f(x, \alpha)$ have no envelope in the neighborhood of (x_0, y_0), except a point envelope which occurs wherever $m \geqq 1$; but if

$$4) \qquad f_{\alpha^{n+1}, x^m}(x_0, \alpha_0) = 0,$$

the family $y = f(x, \alpha)$ has an envelope composed of one or more curves through the point (x_0, y_0); also, whenever $m \geqq 1$, a point envelope at that point.

When $m = 0$, the notation $i = 0, 1, 2, \ldots, m - 1$ is meaningless and 1) must be replaced by $f_\alpha(x_0, \alpha) \neq 0$.

7. *Examples.* p. 86.

PART II. THE IMPLICIT CASE.

8. *Statement of the Problem.* p. 88.

9. *A Sufficient Condition for an Envelope.* p. 88.

If $F(x, y, \alpha)$ is analytic at (x_0, y_0, α_0), and if

$$1) \quad F(x_0, y_0, \alpha_0) = 0, \qquad 2) \quad F_\alpha(x_0, y_0, \alpha_0) = 0,$$

$$3) \quad \frac{D(F, F_\alpha)}{D(x, y)} \equiv \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial F_\alpha}{\partial x} & \frac{\partial F_\alpha}{\partial y} \end{vmatrix} \neq 0 \text{ at } (x_0, y_0, \alpha_0),$$

$$4) \quad F_{\alpha,\alpha}(x_0, y_0, \alpha_0) \neq 0;$$

then the curves $F(x, y, \alpha) = 0$ have an envelope $x = \phi(\alpha)$, $y = \psi(\alpha)$, which consists, in the neighborhood of (x_0, y_0, α_0), of a single analytic curve through this point.

10. *Conditions for Double Points.*

Theorem I. p. 90. Necessary and sufficient conditions that the locus of $F(x, y) = 0$, where $F(x, y)$ is analytic at (x_0, y_0), consist, in the neighborhood of this point, of two analytic branches through (x_0, y_0) with distinct slopes at this point, are that, at (x_0, y_0),

$$1) \quad F = 0, \qquad 2) \quad F_x = 0, \qquad 3) \quad F_y = 0, \qquad 4) \quad F_{xx} F_{yy} - F^2_{xy} \neq 0.$$

Corollary. p. 92. If $F(x, y)$ is real when x and y are real, necessary and sufficient conditions that $F(x, y) = 0$ define at (x_0, y_0) two analytic curves with distinct slopes which are *a*) real, *b*) imaginary, are conditions 1), 2), 3) above, with 4) replaced by

$$4a) \quad F_{xx} F_{yy} - F^2_{xy} \quad 0, \qquad 4b) \quad F_{xx} F_{yy} - F^2_{xy} > 0,$$

respectively.

Theorem II. p. 92. Necessary and sufficient conditions that the locus of $F(x, y) = 0$, where $F(x, y)$ is analytic at (x_0, y_0), consist, in the neighborhood of this point, of two branches through (x_0, y_0) with coincident slopes at this point, are that, at (x_0, y_0),

$$1) \quad F = 0, \qquad 2) \quad F_x = 0, \qquad 3) \quad F_y = 0, \qquad 4) \quad F_{xx} F_{yy} - F^2_{xy} = 0,$$

while 5) F_{xx}, F_{xy}, F_{yy}

do not all vanish. These branches may be connected or not.

11. *The Nodal Locus.*

Theorem I. p. 94. Necessary conditions that a given locus $x = \mu(\alpha)$, $y = \nu(\alpha)$, where $|\mu'(\alpha)| + |\nu'(\alpha)| \neq 0$, be a nodal locus of the curves $F(x, y, \alpha) = 0$; are that the relations

$$1) \quad F \equiv 0, \qquad 2) \quad F_x \equiv 0, \qquad 3) \quad F_y \equiv 0, \qquad 4) \quad F_\alpha \equiv 0,$$

$$5) \quad F_{\alpha x} \neq 0 \text{ or } F_{\alpha y} \neq 0, \qquad 6) \quad F_{xx} F_{yy} - F^2_{xy} \neq 0,$$

$$7)\qquad \begin{vmatrix} F_{xx} & F_{xy} & F_{x\alpha} \\ F_{yx} & F_{yy} & F_{y\alpha} \\ F_{\alpha x} & F_{\alpha y} & F_{\alpha\alpha} \end{vmatrix} \equiv 0$$

be satisfied along this locus.

Theorem II. p. 95. Sufficient conditions that the given locus $x = \mu(\alpha)$, $y = \nu(\alpha)$, where $|\mu'(\alpha)| + |\nu'(\alpha)| \not\equiv 0$, be a nodal locus of the curves $F(x, y, \alpha) = 0$, are that the relations

1) $F \equiv 0$, 2) $F_x \equiv 0$, 3) $F_y \equiv 0$, 4) $F_{xx} F_{yy} - F^2_{xy} \neq 0$

be satisfied along this locus.

Theorem III. p. 95. If, at (x_0, y_0), 1) $F_x = 0$, 2) $F_y = 0$, 3) $F_{xx} F_{yy} - F^2_{xy} \neq 0$, then 4) $F_x(x, y, \alpha) = 0$ and 5) $F_y(x, y, \alpha) = 0$ define $x = \mu(\alpha)$, $y = \nu(\alpha)$, where $\mu(\alpha)$, $\nu(\alpha)$ are analytic at α_0, and where $\mu(\alpha_0) = 0$, $\nu(\alpha_0) = 0$.

If, furthermore, 6) $F(\mu, \nu, \alpha) \equiv 0$, 7) $F_{\alpha x}(\mu, \nu, \alpha) \not\equiv 0$, or $F_{\alpha y}(\mu, \nu, \alpha) \not\equiv 0$, then $|\mu'(\alpha)| + |\nu'(\alpha)| \not\equiv 0$, and $x = \mu(\alpha), y = \nu(\alpha)$ define, in the neighborhood of (x_0, y_0, α_0), a nodal locus of the curves $F(x, y, \alpha) = 0$.

Theorem IV. p. 96. A necessary and sufficient condition that a given nodal locus $x = \mu(\alpha)$, $y = \nu(\alpha)$, where $|\mu'(\alpha)| + |\nu'(\alpha)| \not\equiv 0$, of the curves $F(x, y, \alpha) = 0$ be an envelope of one of the branches at the node, is that $F_{\alpha\alpha}(\mu, \nu, \alpha) \equiv 0$.

12. *Corollary to Theorems I–IV.* p. 97. If $F(x, y, \alpha)$ is real when x, y, α are real, theorems I — IV read for *a*) nodes, *b*) conjugate points, according as we replace the condition $F_{xx} F_{yy} - F^2_{xy} \neq 0$ by *a*) $F_{xx} F_{yy} - F^2_{xy} < 0$, *b*) $F_{xx} F_{yy} - F^2_{xy} > 0$, respectively.

13. *The Locus of Double Points with Coincident Tangents.*

Theorem I. p. 98. Necessary conditions that a given locus $x = \mu(\alpha)$, $y = \nu(\alpha)$, where $|\mu'(\alpha)| + |\nu'(\alpha)| \not\equiv 0$, be a locus of double points with coincident tangents for the curves $F(x, y, \alpha) = 0$, are that 1) $F \equiv 0$, 2) $F_x \equiv 0$, 3) $F_y \equiv 0$, 4) $F_\alpha \equiv 0$, 5) $F_{xx} : F_{xy} : F_{x\alpha} \equiv F_{yx} : F_{yy} : F_{y\alpha} \equiv F_{\alpha x} : F_{\alpha y} : F_{\alpha\alpha}$, along this locus, while 6) F_{xx}, F_{xy}, F_{yy} do not all vanish simultaneously at any point of the given locus.

Theorem II. p. 99. Sufficient conditions that a given locus $x = \mu(\alpha)$, $y = \nu(\alpha)$, where $|\mu'(\alpha)| + |\nu'(\alpha)| \not\equiv 0$, be a locus of double points with coincident tangents for the curves $F(x, y, \alpha) = 0$, are that 1) $F \equiv 0$, 2) $F_x \equiv 0$, 3) $F_y \equiv 0$, 4) $F_{xx} F_{yy} - F^2_{xy} \equiv 0$, along the locus, while 5) F_{xx}, F_{xy}, F_{yy} do not all vanish simultaneously at any point of the given locus.

Theorem III. p. 99. A necessary and sufficient condition that a given locus of double points with coincident tangents, $x = \mu(\alpha)$, $y = \nu(\alpha)$, where $|\mu'(\alpha)| + |\nu'(\alpha)| \not\equiv 0$, of the curves $F(x, y, \alpha) = 0$, be an envelope of these curves is that $F_{\alpha\alpha}(\mu, \nu, \alpha) \equiv 0$.

14. **Examples.** p. 100.

The purpose of this paper is to discuss the conditions under which a family of plane curves, dependent upon a single parameter, will possess an envelope. The discussion is exhaustive for the case that the equation of the curves of the family is given in the explicit form $y = f(x, a)$, $f(x, a)$ being analytic in x, a. It may be noted at the outset that the whole treatment is applicable whether the variables are real or complex, unless otherwise specifically stated. The first part of the paper deals with the equation in the solved form:

$$y = f(x, a),$$

and the second part with the equation in the unsolved form:

$$F(x, y, a) = 0.$$

Part I. The Explicit Case.

1. **Definition of an Envelope.** Let a one-parameter family of plane curves be defined by the equation:

$$(1) \qquad y = f(x, a);$$

where $f(x, a)$ is an analytic function of the two independent variables x and a in the neighborhood of a point (x_0, a_0), i. e., for

$$|x - x_0| < h, \quad |a - a_0| < k;$$

and where

$$f_a(x, a) \neq 0$$

in this neighborhood.

If there exists a curve E, defined by the equations:

$$(2) \qquad x = \phi(a), \quad y = \psi(a),$$

where $\phi(a)$ and $\psi(a)$ are analytic functions of a in the region $|a - a_0| < k' \leqq k$, and where $|\phi'(a)| + |\psi'(a)| > 0$* for a in the same region, and if, for an arbitrary value a' of a in the neighborhood of a_0, the curve $y = f(x, a')$ is tangent to the curve E at the point a', i. e., at the point $x' = \phi(a')$, $y' = \psi(a')$, then the curve E is said to be an envelope of the curves of the family.

2. **Necessary Conditions.** For any $a = a'$ in the neighborhood considered, the point a' of the envelope must be a point of the curve

* This requirement excludes the so-called point-envelopes.

$y = f(x, a')$, and furthermore the slopes of these two curves at this point must be the same ; these facts lead to the two identities in a :

$$(3) \qquad \psi(a) - f[\phi(a), a] \equiv 0,$$

$$(4) \qquad \psi'(a) - \phi'(a) f_x [\phi(a), a] \equiv 0.$$

From differentiation of (3) with respect to a there results the identity :

$$(5) \qquad \psi'(a) - \phi'(a) f_x[\phi(a), a] - f_a[\phi(a), a] \equiv 0,$$

which, subtracted from (4), gives

$$(6) \qquad f_a[\phi(a), a] \equiv 0.$$

The inference is, then, that if an envelope as defined above exists, the two functions $x = \phi(a)$ and $y = \psi(a)$ must satisfy the equations :

$$(7) \qquad y - f(x, a) = 0,$$

$$(8) \qquad f_a(x, a) = 0,$$

which constitute, therefore, necessary conditions.

3. Sufficient Conditions. A set of sufficient conditions may be stated and proved as follows :

THEOREM. *If*

1) $f(x, a)$ *is an analytic function of* x *and* a *in the region*

$$|x - x_0| < h, \quad |a - a_0| < k;$$

2*a*) $$f_a(x, a) \neq 0,$$

in the same region ; but

2*b*) $$f_a(x_0, a_0) = 0 ;$$

3) $$f_{a,x}(x_0, a_0) \neq 0 ;$$

4) $$f_{a,a}(x_0, a_0) \neq 0 ;$$

then the one parameter family of curves defined by $y = f(x, a)$ *has an envelope* E *which is determined by the two equations :*

a) $$f_a(x, a) = 0,$$

b) $$y - f(x, a) = 0.$$

Proof. From the continuity of the functions, conditions 3) and 4) show that $f_{a,x}(x, a) \neq 0$ and $f_{a,a}(x, a) \neq 0$ throughout a certain neighborhood of (x_0, a_0) such as $|x - x_0| < h' \leqq h$, $|a - a_0| < k' \leqq k$. Then by virtue of conditions 2*b*) and 3), equation *a*) may be solved for x, by the ordinary Implicit Function Theorem,* giving $x = \omega(a)$, where $\omega(a)$ is analytic in the region $|a - a_0| < k'' \leqq k'$ and reduces to x_0 for $a = a_0$. Substitution of $\omega(a)$ for x in *b*) yields $y = \chi(a)$, where $\chi(a)$ is analytic in the same region and reduces to y_0 for $a = a_0$.

These functions,

$$x = \omega(a), \quad y = \chi(a),$$

satisfy the two equations *a*) and *b*); thus

$$(9) \qquad f_a[\omega(a), a] \equiv 0, \quad \chi(a) - f[\omega(a), a] \equiv 0,$$

two identities in a for $|a - a_0| < k''$. Differentiating the first of these we find that

$$(10) \qquad f_{a,x}[\omega(a), a]\, \omega'(a) + f_{a,a}[\omega(a), a] \equiv 0,$$

from which, together with 3) and 4), it is clear that

$$(11) \qquad \omega'(a) \equiv -\frac{f_{a,a}(x,a)}{f_{a,x}(x,a)} \neq 0, \neq \infty, \qquad x = \omega(a).$$

The slope of the curve $x = \omega(a)$, $y = \chi(a)$ at any point a is given by the relation:

$$(12) \qquad \frac{dy}{dx} = \frac{\chi'(a)}{\omega'(a)}.$$

The slope of the curve of the family $y = f(x, a)$, for this value of a and at any of its points, is determined by the equation:

$$(13) \qquad \frac{dy}{dx} = f_x(x,a).$$

But the second identity of (9) yields on differentiation:

$$(14) \qquad \chi'(a) - f_x[\omega(a), a]\, \omega'(a) - f_a[\omega(a), a] \equiv 0,$$

which, with the first of identities (9), shows that

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* Osgood: *Lehrbuch der Funktionentheorie*, p. 48, vol. 1, and p. 345. The proof given for algebraic equations holds for the more general type here considered.

$$(15) \qquad \frac{\chi'(a)}{\omega'(a)} \equiv f_x(x, a), \qquad x = \omega(a),$$

thus proving the slopes identical at the arbitrary point $x = \omega(a)$, $y = \chi(a)$.

Since, by (11), $\omega'(a) \neq 0$, it follows that the condition $|\omega'(a)| + |\chi'(a)| > 0$ is satisfied and the curve above determined meets all the requirements of our definition of an envelope.

4. Some Limitations of these Conditions. The conditions of the foregoing article are sufficient to give an envelope which, at and near a given point a_0, is composed of a single analytic branch. They are not necessary even for this restricted case. Let us consider two examples.

Example 1. Does the family of curves

$$y = (x - a)^2$$

have an envelope which is smooth in and about the point $a = a_0 = 0$, $x = x_0 = 0$? Here

$$f_a(x, a) \equiv -2(x - a) \not\equiv 0, \qquad f_a(0, 0) = 0;$$

$$f_{a,x}(0, 0) = -2 \neq 0, \qquad f_{a,a}(0,0) = 2 \neq 0.$$

These being satisfied, it is assured that the curves in question have an envelope smooth at the point $x_0 = 0$, $a_0 = 0$, and that this envelope is obtained by solving the equations:

$$a) \qquad f_a(x, a) \equiv -2(x - a) = 0,$$

$$b) \qquad y - f(x, a) \equiv y - (x - a)^2 = 0,$$

for x and y in terms of a. The solution is immediate and the envelope of the system is

$$y = 0, \qquad (x = a).$$

Example 2. Does the family of curves

$$y = (x - a)^3$$

have an envelope which is smooth in and about the point $a = a_0 = 0$, $x = x_0 = 0$? Conditions 1) and 2) are fulfilled just as before but 3) and 4) are violated, for

$$f_{a,x}(x_0, a_0) = -6(x_0 - a_0) = 0, \quad f_{a,a}(x_0, a_0) = 6(x_0 - a_0) = 0.$$

Geometrically it is evident that this family of curves also has an envelope which is quite as smooth as that of the curves of Example 1, being, indeed, identical with it. In fact, the family

$$y = (x - a)^n, \quad n = 3, 4, \cdots$$

has $y = 0$ as an envelope though conditions 3) and 4) both fail.

Since some of the hypotheses of the theorem fail in cases where envelopes are known to exist, it is clear that the theorem contains hypotheses that are unnecessarily restrictive.

5. Weierstrass's Theorem on Implicit Functions. The narrowness of the theorem of §3 arises from the weakness of the Implicit Function Theorem employed for the solution of the equation $f_a(x, a) = 0$. This theorem gives conditions sufficient, but so restrictive as to leave untouched a number of cases which it is essential to be able to treat. As a means of handling these cases use is made of the Implicit Function Theorem of Weierstrass, which may be stated as follows:

WEIERSTRASS'S THEOREM. *Let $F(x, y)$ be a function of the two independent variables x and y, analytic in the neighborhood of a point (x_0, y_0); let*

$$F(x_0, y) \not\equiv 0,$$

and let

$$F(x_0, y_0) = 0;$$

then the function $F(x, y)$ may be written thus:

$$F(x, y) \equiv P(x, y) \cdot H(x, y),$$

wnere

$$P(x, y) \equiv (y - y_0)^n + A_1(x)(y - y_0)^{n-1} + \cdots + A_n(x),$$

*the $A_i(x)$ being analytic functions of x in the neighborhood of x_0 and vanishing for $x = x_0$, and $H(x, y)$ being an analytic function of x and y in the neighborhood of (x_0, y_0), not vanishing at (x_0, y_0).**

The inference is, therefore, that the equation $F(x, y) = 0$ has n, and only n, roots y as functions of x which approach y_0 as x approaches x_0, and that these roots are the n roots of $P(x, y) = 0$. Attention may thus be

* For the proof of this theorem, the reader is referred to Goursat: *Cours d'analyse*, vol. 2, pp. 280–285. See also Picard, *Traité d'analyse*, vol. 2, p. 243; Bliss, *Bull. Amer. Math. Soc.*, vol. 16, p. 356 (April, 1910); MacMillan, *ibid.*, vol. 17, p. 116 (Dec. 1910).

restricted in a study of the behavior of the roots of $F(x, y) = 0$ in the neighborhood in question to a consideration of the roots of the simpler equation $P(x, y) = 0$.

To the discussion of this equation the following considerations are essential.

Definition. The function

$$P(x, y) \equiv y^n + A_1(x)y^{n-1} + A_2(x)y^{n-2} + \cdots + A_n(x),$$

where $A_i(x)$ is analytic at $x = x_0$ and $A_i(x_0) = 0$, is *reducible* at the point $(x = x_0, y = y_0 = 0)$ if

$$P(x,y) \equiv [y^m + B_1(x)y^{m-1} + \cdots + B_m(x)][y^l + C_1(x)y^{l-1} + \cdots + C_l(x)],$$

where $m > 0$, $l > 0$, and $B_j(x)$ and $C_k(x)$ satisfy the same conditions as $A_i(x)$. Otherwise it is *irreducible*.

THEOREM. *A necessary and sufficient condition that $P(x, y)$ be irreducible at the point $(x_0, y_0 = 0)$ is that, no matter how small a neighborhood of the point x_0 one consider, the Riemann surface for the function y of x defined by $P(x, y) = 0$ spread out over this neighborhood consist of a single piece.*

Proof. 1. *The condition is sufficient.* For if $P(x, y)$ be reducible the Riemann surface for y as a function of x defined by $P(x, y) = 0$ will, in the neighborhood of x_0, be composed of the separate Riemann surfaces corresponding to the separate irreducible factors.*

2. *The condition is necessary.* The equation $P(x, y) = 0$ defines y as an n-valued function of x in the neighborhood of $x = x_0$; let the branches be denoted by $f_1(x), f_2(x), \cdots, f_n(x)$. Assume that the Riemann surface breaks up and let one of the pieces be a λ-sheeted piece, $\lambda < n$, to which are assigned the branches $f_1(x), \cdots, f_\lambda(x)$. Sever the surface by a cut running out indefinitely from the point $x = x_0$, and let the description of a closed circuit around $x = x_0$ by the independent variable cause $f_1(x)$ to go over into $f_2(x)$, $f_2(x)$ into $f_3(x)$, and so on, $f_\lambda(x)$ going over into $f_1(x)$. Consider then the function

$$\bar{P}(x, y) \equiv [y - f_1(x)][y - f_2(x)] \cdots [y - f_\lambda(x)]$$
$$\equiv y^\lambda + \beta_1(x)\, y^{\lambda-1} + \cdots + \beta_\lambda(x).$$

The functions $\beta_i(x)$ are integral symmetric functions of the f's, and con-

* See, in this connection, Osgood: *Lehrbuch der Funktionentheorie*, vol. 1, p. 357.

sequently are functions of x, single-valued and analytic throughout the unsevered neighborhood of $x = x_0$, except possibly at x_0 itself. Moreover $\beta_i(x)$ is finite and continuous in this region, and $\beta_i(x_0) = 0$. Hence $\beta_i(x)$ is analytic throughout the unsevered neighborhood of $x = x_0$ inclusive of the point itself.* For if a function is analytic throughout the neighborhood of a point, with the possible exception of the point itself, and if it remains finite in the neighborhood, then it approaches a limit as the variable approaches the point; furthermore if this limit is assigned as the value of the function at the point, then the function is analytic at this point also. These conditions are fulfilled in the present case. The function $\bar{P}(x, y)$ is therefore irreducible at the point $x = x_0$ (by part 1 of this proof).

Hence it is seen that $P(x, y)$ admits as a factor an irreducible polynomial, $\bar{P}(x, y)$, of the same type. This contradicts the hypothesis that $P(x, y)$ is irreducible, and the theorem is proved.

Equating to zero each of the irreducible factors of $P(x, y)$ we have y defined as a function of x, single or multiple valued as the case may be. In case the function corresponding to a given irreducible factor is multiple valued, the branches are all connected in a single cycle at x_0. In this case all the values of y corresponding to an arbitrary $x \neq x_0$ in a suitably restricted neighborhood of x_0 are distinct; for the discriminant of the irreducible factor is a function of x, analytic in the neighborhood of the point x_0 and vanishing there, but not vanishing identically. It has, therefore, no second root in the vicinity of x_0. Furthermore unless this function y of x is a constant, the inverse function exists and x can be expressed as a single valued function of y in the neighborhood of y_0, or as a multiple valued function whose branches are connected in cycle at y_0. There will, of course, be as many such functions x as there are irreducible factors in $P(x, y)$.

6. Application of the Theorem of the Preceding Article. A necessary condition for an envelope of the curves $y = f(x, a)$ has been seen to consist in the equation

$$f_a(x, a) = 0.$$

We now proceed to inquire how far this condition is sufficient.

The function $f_a(x, a)$ may contain certain easily detected factors which are extraneous for our present purposes, factors, namely, of either or both of the forms

$$(x - x_0)^m, \qquad (a - a_0)^n,$$

* See Osgood: *Funktionentheorie*, vol. 1, p. 262, Riemannscher Satz.

where m and n are positive integers. If $f_{a^{n+1}}(x, a_0) \neq 0$, but all the preceding a-derivatives of $f_a(x, a)$ —if there are any—vanish identically for $a = a_0$, then $f_a(x, a)$ contains the factor $(a - a_0)^n$ but not the factor $(a - a_0)^{n+1}$. If $f_{a,x^m}(x_0, a) \neq 0$, but all the preceding x-derivatives of $f_a(x, a)$—if there are any—vanish identically for $x = x_0$, then $f_a(x, a)$ contains the factor $(x - x_0)^m$ but not the factor $(x - x_0)^{m+1}$. Hence we may write

$$f_a(x, a) \equiv (a - a_0)^n (x - x_0)^m F(x, a),$$

where m, n are determined as just indicated, and where

$$F(x, a_0) \neq 0, \qquad F(x_0, a) \neq 0.$$

The solutions of the equation

$$f_a(x, a) = 0$$

are the solutions of the three separate equations,

a) $$a - a_0 = 0,$$

b) $$x - x_0 = 0,$$

c) $$F(x, a) = 0.$$

Equation a) does not define x as a function of a; hence the factor $(a - a_0)^n$ gives rise to no envelope.

Equation b) gives $x = x_0$. Writing $f_a(x, a)$ as

$$f_a(x, a) \equiv (x - x_0)^m (a - a_0)^n F(x, a)$$

$$\equiv (x - x_0)^m K(x, a),$$

we have

$$y = f(x, a) = (x - x_0)^m \int_{a_0}^{a} K(x, a) da + L(x);$$

then $$y_0 = f(x_0, a) = L(x_0),$$

and all the curves, $y = f(x, a)$, are seen to pass through the fixed point (x_0, y_0). Hence the factor $(x - x_0)^m$ gives rise to the so-called *point envelope*.

Equation c) presents two cases according as

$$F(x_0, a_0) \neq 0, \quad \text{or} \quad F(x_0, a_0) = 0.$$

If $F(x_0, a_0) \neq 0$, then $F(x, a)$ is the function $H(x, a)$ obtained by applying Weierstrass's Theorem to the function

$$(x - x_0)^m F(x, a).$$

Hence, if $F(x_0, a_0) \neq 0$, the factor $F(x, a)$ gives rise to no envelope.

If $F(x_0, a_0) = 0$, Weierstrass's Theorem enables us to solve the equation

$$c) \qquad F(x, a) = 0$$

for x as a function of a, $x = \phi(a)$, in the neighborhood of $a = a_0$. The functions

$$x = \phi(a), \qquad y = f[\phi(a), a] = \psi(a),$$

define an envelope of the curves

$$y = f(x, a)$$

in the region in question. These statements we now proceed to prove.

By Weierstrass's Theorem, we may write

$$F(x, a) \equiv \Omega(x, a)\, H(x, a),$$

where $H(x, a)$ is analytic in the neighborhood of (x_0, a_0) and $H(x_0, a_0) \neq 0$. The x-roots of $F(x, a) = 0$ in the neighborhood of $a = a_0$ are all given as the roots of the equation

$$\Omega(x, a) = 0.$$

Let $\Omega_i(x, a)$ be a factor of $\Omega(x, a)$, irreducible in the neighborhood of a_0; it may be written

$$\Omega_i(x, a) \equiv (x - x_0)^l + A_1(a)(x - x_0)^{l-1} + \cdots + A_l(a),$$

where $A_j(a)$ is analytic at a_0, $A_j(a_0) = 0$, and $A_l(a) \neq 0$. Then

$$\Omega_i(x, a) = 0$$

defines x as an l-valued function of a,

$$x = \phi(a),$$

whose branches are all distinct except at a_0, where they are all connected in a single cycle. Cut each leaf of the Riemann surface along a ray emanating

from a_0, and denote the branches of $\phi(a)$ corresponding to the severed leaves by $\phi_1(a), \phi_2(a), \phi_3(a), \cdots, \phi_l(a)$. The substitution of these functions in the equation

$$y = f(x, a)$$

gives

$$y = f[\phi_j(a), a] = \psi_j(a) ;$$

thus we obtain pairs of functions

$$x = \phi_j(a), \qquad y = \psi_j(a),$$

such that in the region in question no two branches $\phi_i(a), \phi_j(a)$ are the same.

In the neighborhood of any point $a = a_1 \neq a_0$ in a suitably restricted vicinity of a_0, the functions,

$$x = \phi_j(a), \qquad y = \psi_j(a),$$

define an envelope of the family $y = f(x, a)$.

For, from the character of $F(x, a)$, the function $\phi_j(a)$ is non-constant, and $\phi_j'(a) \neq 0$; hence the condition

$$|\phi_j'(a)| + |\psi_j'(a)| > 0$$

of the definition (§2) is fulfilled at any point of the region in question. The proof that the slopes of the curves

$$x = \phi_j(a), \qquad y = \psi_j(a),$$

and

$$y = f(x, a)$$

are identical at an arbitrary point of the neighborhood of a_1 is exactly that used in establishing the theorem of §3. Thus the functions $x = \phi_j(a)$, $y = \psi_j(a)$ meet all the requirements of the definition of §2 and the statement is established.

To the irreducible factor $\Omega_i(x, a)$ there corresponds therefore an envelope consisting of a single branch analytic at a_0, or of several branches distinct and analytic in the neighborhood of a_0 except at the point itself and meeting at the point. To the function $\Omega(x, a)$ will correspond, of course, one envelope of this type for each of its distinct irreducible factors. These envelopes will, in general, be distinct; see §7, example 4, for an exception.

The results of the foregoing discussion are brought together in the follow-

ing fundamental theorem, which summarizes the facts concerning envelopes of the family of curves given in the explicit form $y = f(x, a)$.

Fundamental Theorem. *Given a one-parameter family of curves $y = f(x, a)$, where $f(x, a)$ is an analytic function of x and a in the neighborhood of (x_0, a_0), such that*

$$1) \qquad f_{a,x^i}(x_0, a) \equiv 0, \qquad i = 0, 1, 2, \cdots, m-1,$$

but

$$f_{a,x^m}(x_0, a) \not\equiv 0,$$

$$2) \qquad f_{a^j}(x, a_0) \equiv 0, \qquad j = 1, 2, \cdots, n,$$

but

$$f_{a^{n+1}}(x, a_0) \not\equiv 0;$$

then, if

$$3) \qquad f_{a^{n+1},x^m}(x_0, a_0) \neq 0,$$

the curves $y = f(x, a)$ have no envelope in the neighborhood of (x_0, y_0), except a point envelope which occurs whenever $m \geqq 1$; but, if

$$3') \qquad f_{a^{n+1},x^m}(x_0, a_0) = 0,$$

the family $y = f(x, a)$ has an envelope composed of one or more curves through the point (x_0, y_0); also, whenever $m \geqq 1$, a point envelope at that point.

When $m = 0$, the notation $i = 0, 1, 2, \cdots, m-1$ is meaningless and 1) must be replaced by $f_a(x_0, a) \not\equiv 0$.

7. Examples. *Example* 1. Examine for an envelope in the neighborhood of the point $x_0 = 0$, $a_0 = 0$, the curves:

$$y = f(x, a) \equiv 4a^5 - 5a^4x + 4a^5x^2 - 5a^4x^3 - x^9.$$

Here $f(x, a)$ is analytic at $(0, 0)$, and

$$f_a(x, a) \equiv 20\, a^3(a - x)(1 + x^2),$$

so that $f_a(x_0, a_0) = 0$ while $f_a(x_0, a) \not\equiv 0$. The conditions of the Fundamental Theorem are fulfilled, $n = 3$, $m = 0$, and

$$\Omega\,(x, a) \equiv (a - x), \qquad H(x, a) \equiv 20(1 + x^2).$$

Thus we obtain the envelope

$$x = a, \qquad y = -\,a^5(1 + a + a^2)(1 - a + a^2),$$

consisting of one analytic branch through $(0,0)$; the equation in Cartesian form is

$$x^9 + x^7 + x^5 + y = 0.$$

Example 2. Examine for an envelope in the neighborhood of $x_0 = 0$, $a_0 = 0$, the curves:

$$y = f(x, a) \equiv 20x^4a^3 + x^7 + 15a^4x^2 + 15a^4x^4 + 12x^2a^5.$$

Here $$f_a(x, a) \equiv 60a^2x^2\,(x^2 + a)(1 + a),$$

and we have the conditions of the Fundamental Theorem with $m = 2$ and $n = 2$. Thus we obtain a point envelope at the origin, and an envelope which consists of a curve through $(0,0)$ given by

$$x = \pm\sqrt{-a}, \qquad y = 5a^5 + 3a^6 \mp a^3\sqrt{-a},$$

or in Cartesian form:

$$y = x^7(1 - 5x^3 + 3x^5).$$

Example 3. Examine for an envelope in the neighborhood of $x_0 = 0$, $a_0 = 0$, the curves:

$$y = f(x, a) \equiv x^2\left[ax^4 + 2a^2x^3 + \left(\frac{4}{3}\,a - \frac{1}{2}\right)a^2x^2 - \frac{4}{3}\,a^3x - a^4\right].$$

Here $$f_a(x, a) \equiv x^2\left[x^4 + 4ax^3 + (4a^2 - a)x^2 - 4a^2x - 4a^3\right]$$
$$\equiv x^2(x^2 - a)(\,x + 2a)^2.$$

The envelope consists of the curves

$$x = a^{\frac{1}{2}}, \quad y = \frac{a^4}{6}\left[3 + 4a^{\frac{1}{2}} + 2a\right],$$

and $$x = -\,2a, \; y = \frac{4}{3}a^6\left[112a - 17\right];$$

the origin is a point envelope.

Example 4. Examine for an envelope in the neighborhood of $x_0 = 0$, $a_0 = 0$, the curves:

$$y = f(x, a) \equiv (x - a)^2\,(x - 2a)^2.$$

Here $$f_a(x, a) \equiv 2(x - a)(x - 2a)(4a - 3x),$$

and
$$\Omega(x, a) \equiv (x - a)(x - 2a)(x - \tfrac{4}{3}a).$$

The Riemann surface consists of three distinct pieces, each composed of a single sheet. The envelope is made up of the loci

$$x = a, \quad y = 0; \quad x = 2a, \quad y = 0;$$

and
$$x = \frac{4}{3}a, \quad y = \frac{4}{81}a^4.$$

This example illustrates how two different irreducible factors of $\Omega(x, a)$ may give rise to the same envelope in the neighborhood of a point (x_0, y_0).

Part II. The Implicit Case.

8. Statement of the Problem. Attention will now be directed to the case in which the family of plane curves is defined by an equation in the implicit form:

$$F(x, y, a) = 0,$$

where $F(x, y, a)$ is an analytic function of x, y, a and where $F_a(x, y, a) \not\equiv 0$. The definition of the envelope is the same as in §1. Two cases will be considered here: a) that in which the curves of the family have no multiple points in the region in question, b) that in which these curves have multiple points of order two only.

9. A Sufficient Condition for an Envelope. In Case a) the following theorem gives sufficient conditions for the existence of an envelope.

Theorem. *If $F(x, y, a)$ is a function, analytic in the three independent variables x, y, and a, at the point (x_0, y_0, a_0), and if*

1) $$F(x_0, y_0, a_0) = 0 \text{ and } F_a(x_0, y_0, a_0) = 0,$$

2) $$\frac{D(F, F_a)}{D(x, y)} \equiv \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial F_a}{\partial x} & \frac{\partial F_a}{\partial y} \end{vmatrix} \neq 0 \text{ at } (x_0, y_0, a_0),$$

3) $$F_{\alpha\alpha}(x_0, y_0\ \alpha_0) \neq 0;$$

then the curves $F(x, y, \alpha) = 0$ *have an envelope* $x = \phi(\alpha)$, $y = \psi(\alpha)$, *which consists, in the neighborhood of the point* (x_0, y_0, α_0), *of a single analytic curve.*

Proof: The two functions $F(x, y, \alpha)$ and $F_\alpha(x, y, \alpha)$ satisfy the conditions for obtaining x and y from the two simultaneous equations

$$F(x, y, \alpha) = 0, \quad F_\alpha(x, y, \alpha) = 0$$

as given in the ordinary theorem on Implicit Functions.* The solution of the equations gives rise to two analytic functions,

$$x = \phi(\alpha), \quad y = \psi(\alpha),$$

such that $F[\phi(\alpha), \psi(\alpha), \alpha] \equiv 0$ and $F_\alpha[\phi(\alpha), \psi(\alpha), \alpha] \equiv 0$, and that $\phi(\alpha_0) = x_0$ and $\psi(\alpha_0) = y_0$. These two functions $x = \phi(\alpha)$, $y = \psi(\alpha)$ exhaust all the systems of values (x, y, α) in the neighborhood of (x_0, y_0, α_0) for which the functions $F(x, y, \alpha)$ and $F_\alpha(x, y, \alpha)$ vanish simultaneously. Hence but a single curve is determined. Furthermore, $\phi(\alpha)$ and $\psi(\alpha)$ are not both constant; for differentiate the identity $F_\alpha(\phi, \psi, \alpha) \equiv 0$, thus obtaining

$$F_{\alpha x}\phi'(\alpha) + F_{\alpha y}\psi'(\alpha) + F_{\alpha\alpha} \equiv 0.$$

Then, if $\phi(\alpha)$ and $\psi(\alpha)$ were constant, $\phi'(\alpha) \equiv \psi'(\alpha) \equiv 0$, and $F_{\alpha\alpha}[\phi(\alpha), \psi(\alpha), \alpha] \equiv 0$ and, in particular, $F_{\alpha\alpha}[\phi(\alpha_0), \psi(\alpha_0), \alpha_0] = 0$, in contradiction of condition 3). Therefore $|\phi'(\alpha)| + |\psi'(\alpha)| > 0$.

It remains merely to show that, at the common point (x_0, y_0, α_0), the curves $F(x, y, \alpha_0) = 0$ and $x = \phi(\alpha)$, $y = \psi(\alpha)$ have the same slope. Differentiating the identity $F(\phi, \psi, \alpha) \equiv 0$, we get

$$F_x\phi'(\alpha) + F_y\psi'(\alpha) + F_\alpha \equiv 0.$$

But at the points under consideration $F_\alpha \equiv 0$, and thus the last identity reduces to the form

$$F_x\phi'(\alpha) + F_y\psi'(\alpha) \equiv 0.$$

Now the slope of $F(x, y, \alpha_0) = 0$ is given by the equation $F_x dx + F_y dy = 0$, and F_x, F_y are not both zero, by 2), while the slope of $x = \phi(\alpha)$, $y = \psi(\alpha)$ is given by the equation $\psi'(\alpha)\, dx - \phi'(\alpha)\, dy = 0$, and the above identity is

* See Osgood: *Lehrbuch der Funktionentheorie*, vol. 1, pp. 52–55 and p. 345. The proof for a single equation is readily extended to a simultaneous system.

precisely the necessary and sufficient condition that these two equations, homogeneous in dx and dy, have a common solution other than $(0, 0)$. The theorem therefore is proved. The conditions are *sufficient*, but not *necessary*.

10. Conditions for Double Points. In case $b)$ there will occur, in the pairs of solutions for x and y in terms of a from $F(x, y, a) = 0$ and $F_a(x, y, a) = 0$, not only the envelope, in case there is one, but also the locus of double points. In the complex domain, we distinguish two classes of multiple points of order two, according as the tangents to the curve at the point are distinct or coincident. The first class we call *nodes;* the second class includes *cusps*, *tacnodes*, *points of osculation* and so forth. In the domain of reals, double points with distinct tangents are termed *nodes* or *conjugate points*, according as these tangents are real or imaginary. That we may get simple, definite conditions, we shall make in all the following discussion the restriction on the function $F(x, y, a)$ that it involve no repeated factor. Then we may state the following theorem for a node.

THEOREM I. *Necessary and sufficient conditions that the locus of* $F(x, y) = 0$, *where* $F(x, y)$ *is an analytic function of* x *and* y *in the neighborhood of* (x_0, y_0), *consist, in this neighborhood, of two analytic branches through* (x_0, y_0) *with distinct slopes at this point, are that, at* (x_0, y_0),

$$1)\ F = 0, \quad 2)\ F_x = 0, \quad 3)\ F_y = 0, \quad 4)\ F_{xx}F_{yy} - F_{xy}^2 \neq 0.$$

Proof. There is no loss of generality in taking (x_0, y_0) as $(0,0)$.

1. *The conditions are necessary.* Since $(0,0)$ is a point on the locus, $F(0, 0) = 0$. If either 2) or 3) is violated, the ordinary Implicit Function Theorem shows that the locus is a single analytic curve, contrary to the hypothesis. The function $F(x, y)$ may, therefore, be written in the form:

$$\begin{aligned} F(x, y) &\equiv ax^2 + 2bxy + cy^2 + dx^3 + 3ex^2y + \cdots \\ &\equiv x^2\left(a + 2b\frac{y}{x} + c\frac{y^2}{x^2}\right) + x^3\left(d + 3e\frac{y}{x} + \cdots\right) + \cdots \\ &\equiv x^2 f(x, \lambda), \qquad\qquad \lambda = \frac{y}{x}. \end{aligned}$$

The only solution of $F(0, y) = 0$ is $y = 0$. Consider a particular branch of the locus through $(0, 0)$; its slope is

$$\lim_{x \doteq 0} \frac{y}{x} = \lim_{x \doteq 0} \lambda,$$

y and λ being considered as functions of the independent variable x. The variables x and λ are connected by the equation

$$f(x, \lambda) \equiv a + 2b\lambda + c\lambda^2 + x(d + 3e\lambda + \cdots) + \cdots = 0.$$

Moreover lim $x = 0$; consequently the limiting value of λ satisfies the equation

$$f(0, \lambda) \equiv a + 2b\lambda + c\lambda^2 = 0.$$

Since, by hypothesis, the slopes, λ_1, λ_2, at $(0, 0)$ are distinct, $b^2 - ac \neq 0$. But $F_{xx}(0, 0) = 2a$, $F_{xy}(0, 0) = 2b$, $F_{yy}(0, 0) = 2c$; hence the quadratic equation in λ is equivalent to either of the two equations,

$$ax^2 + 2bxy + cy^2 = 0, \text{ and } F_{xx}(dx)^2 + 2F_{xy}\,dxdy + F_{yy}(dy)^2 = 0,$$

and the condition just obtained becomes

$$4) \qquad F_{xx}\,F_{yy} - F_{xy}^2 \neq 0.$$

This establishes the necessary conditions.

2. *The conditions are sufficient.* Because of 1), 2), 3), we have, as above,

$$F(x, y) \equiv ax^2 + 2bxy + cy^2 + dx^3 + 3ex^2y + \cdots$$

$$\equiv x^2 f(x, \lambda), \qquad \lambda = \frac{y}{x},$$

where, because of 4),

$$b^2 - ac \neq 0.$$

The only solution of $F(0, y) = 0$ is $y = 0$. Consider, then, the function

$$f(x, \lambda) \equiv a + 2b\lambda + c\lambda^2 + x(d + 3e\lambda + \cdots) + \cdots$$

$$\equiv A_0(\lambda) + A_1(\lambda) \cdot x + A_2(\lambda) \cdot x^2 + \cdots.$$

The roots, λ_1, λ_2, of $A_0(\lambda) = 0$ are distinct; hence

$$\frac{\partial f(0, \lambda_i)}{\partial \lambda} = 2(b + c\lambda_i) \neq 0, \qquad i = 1, 2.$$

Therefore $$f(x, \lambda) = 0$$

determines $\lambda = \phi_i(x)$, $i = 1, 2$, $[\phi_i(0) = \lambda_i, f(x, \phi_i(x) \equiv 0]$, a pair of functions, each single valued and continuous in the neighborhood of $x = 0$, and having there a continuous first derivative. Then $y = \lambda x$ gives two functions

$$y = x\phi_1(x) = \phi(x), \qquad y = x\phi_2(x) = \psi(x),$$

which define two analytic curves through $(0, 0)$.

The slopes of the curves at $(0, 0)$ are

$$\left.\frac{dy}{dx}\right|_{x=x_0=0} = x_0\phi_i'(x_0) + \phi_i(x_0) = \lambda_i, \qquad i = 1, 2.$$

But $\lambda_1 \neq \lambda_2$; hence the two curves through $(0, 0)$ have there distinct slopes.

The curves $\lambda = \phi_i(x)$ exhaust all points in the neighborhood of $(0, \lambda_i)$ for which $f(x, \lambda)$ vanishes. To show that these curves exhaust all points in the neighborhood of $x = 0$, $\lambda = \lambda$, for which $f(x, \lambda)$ vanishes, proceed as follows. Take $\lambda = \lambda_3$ any point not in the neighborhood of either λ_1 or λ_2. Write $f(x, \lambda)$ as

$$f(x, \lambda) \equiv A_0(\lambda) + \sum_{m=1}^{\infty} A_m(\lambda)x^m.$$

Then $|A_0(\lambda_3)| > G$, G being some fixed positive number; moreover, there exists a positive constant ρ, such that

$$\left|\sum_{m=1}^{\infty} A_m(\lambda_3)\, x^m\right| < G, \qquad \text{for } |x| < \rho.$$

Hence this value pair (x, λ_3) does not cause $f(x, \lambda)$ to vanish. This shows that the complete locus of $F(x, y) = 0$ is represented, in the neighborhood of $(0, 0)$, by the two curves

$$y = \phi(x), \qquad y = \psi(x),$$

and the theorem is proved.

COROLLARY. *If $F(x, y)$ is real when x and y are real, necessary and sufficient conditions that $F(x, y) = 0$ define at (x_0, y_0) two analytic curves with distinct slopes which are a) real, b) imaginary, are conditions* 1), 2), 3) *above, with* 4) *replaced by*

$$4a)\ F_{xx}F_{yy} - F_{xy}^2 < 0, \qquad 4b)\ F_{xx}F_{yy} - F_{xy}^2 > 0,$$

respectively.

THEOREM II. *Necessary and sufficient conditions that the locus of $F(x, y) = 0$, where $F(x, y)$ is an analytic function of x and y in the neighborhood of (x_0, y_0), consist, in this neighborhood, of two branches through (x_0, y_0) with coincident slopes at this point, are that, at (x_0, y_0),*

$$1)\ F = 0, \quad 2)\ F_x = 0, \quad 3)\ F_y = 0, \quad 4)\ F_{xx}F_{yy} - F^2_{xy} = 0,$$

while 5) $$F_{xx}, \quad F_{xy}, \quad F_{yy}$$

do not all vanish. These branches may be connected or not.

Proof. As before let us take (x_0, y_0) as $(0, 0)$.

Necessary Conditions. These are established precisely as in the foregoing theorem.

Sufficient Conditions. Because of 1), 2), 3), we have

$$F(x, y) \equiv ax^2 + 2bxy + cy^2 + dx^3 + 3ex^2y + \cdots,$$

where, because of 4), 5),

$$b^2 - ac = 0, \qquad a, b, c, \text{ not all zero.}$$

Since either $a \neq 0$ or $c \neq 0$, let us assume that $c \neq 0$. Then

$$F(0, y) \not\equiv 0, \quad F(0, 0) = 0, \quad F_{y,y}(0, 0) = 2c \neq 0.$$

Hence we have, by Weierstrass's Theorem,

$$F(x, y) \equiv \Omega(x, y) \cdot H(x, y),$$

where $H(x, y)$ is analytic at $(0, 0)$, $H(0,0) \neq 0$, and where

$$\Omega(x, y) \equiv y^2 + A_1(x)y + A_2(x),$$

$$A_i(x) \text{ analytic at } x = 0, \qquad A_i(0) = 0.$$

The locus of $\Omega(x, y) = 0$ exhausts all points in the neighborhood of $(0, 0)$ for which $F(x, y)$ vanishes. We may write

$$\Omega(x, y) \equiv y^2 + A_1(x)y + A_2(x)$$
$$\equiv \left(y + \frac{A_1 - B}{2}\right)\left(y + \frac{A_1 + B}{2}\right), \qquad B^2 \equiv A_1^2 - 4A_2,$$

where $B^2(x) \not\equiv 0$, because $F(x, y)$ has, by hypothesis, no repeated factor. If

the expansion of $B^2(x)$ about $x = 0$ begins with an even power of x, $B(x)$ is analytic at $x = 0$, $\Omega(x, y)$ is reducible, and $\Omega(x, y) = 0$ defines two branches, analytic at $(0, 0)$. This is the case at a tacnode. If the expansion of $B^2(x)$ about $x = 0$ begins with an odd power of x, $B(x)$ is not analytic at the point, $\Omega(x, y)$ is irreducible, and $\Omega(x, y) = 0$ defines two branches, analytic in the neighborhood of $(0,0)$, but connected at that point.

If $c = 0$, then $a \neq 0$, Weierstrass's Theorem gives

$$\Omega(x, y) \equiv x^2 + B_1(y)\, x + B_2(y),$$

and $\Omega(x, y) = 0$ defines two branches as before.

The slopes, at $(0,0)$, of the locus of $F(x, y) = 0$ are determined, as in Theorem I, by the equation

$$a + 2b\lambda + c\lambda^2 = 0.$$

Since, by hypothesis, $b^2 - ac = 0$, these two slopes are equal, and the theorem is proved.

11. The Nodal Locus. THEOREM I. *A necessary condition that a given locus $x = \mu(a)$, $y = \nu(a)$, where $|\mu'(a)| + |\nu'(a)| \neq 0$, be a nodal locus of the curves of the family $F(x, y, a) = 0$ is that the relations*

$$1)\ F \equiv 0, \quad 2)\ F_x \equiv 0, \quad 3)\ F_y \equiv 0, \quad 4)\ F_a \equiv 0,$$

$$5)\ F_{ax} \not\equiv 0 \ \text{ or } \ F_{ay} \not\equiv 0, \quad 6)\ F_{xx}F_{yy} - F^2_{xy} \neq 0,$$

$$7) \qquad \begin{vmatrix} F_{xx} & F_{xy} & F_{xa} \\ F_{yx} & F_{yy} & F_{ya} \\ F_{ax} & F_{ay} & F_{aa} \end{vmatrix} \equiv 0,$$

be satisfied along this locus.

Proof. Since $x_i = \mu(a_i)$, $y_i = \nu(a_i)$ is a node of the curve a_i, therefore, by the preceding section

$$F = 0, \quad F_x = 0, \quad F_y = 0, \quad F_{xx}F_{yy} - F^2_{xy} \neq 0,$$

at this point. Since these conditions are true at every point of the locus, then

$$1)\ F \equiv 0, \quad 2)\ F_x \equiv 0, \quad 3)\ F_y \equiv 0, \quad 6)\ F_{xx}F_{yy} - F^2_{xy} \neq 0,$$

along the locus $x = \mu(a)$, $y = \nu(a)$. From the first identity,

$$F_x\,dx + F_y\,dy + F_a\,da \equiv 0$$

along the same locus. But since $F_x \equiv 0$ and $F_y \equiv 0$ here, it follows that 4) $F_a(x, y, a) \equiv 0$ along this locus. Differentiating the identities 2), 3), 4), we obtain

$$F_{xx}\mu' + F_{xy}\nu' + F_{xa} \equiv 0,$$
$$F_{yx}\mu' + F_{yy}\nu' + F_{ya} \equiv 0,$$
$$F_{ax}\mu' + F_{ay}\nu' + F_{aa} \equiv 0,$$

along $x = \mu(a)$, $y = \nu(a)$. A necessary condition that these be consistent is

$$7) \qquad \begin{vmatrix} F_{xx} & F_{xy} & F_{xa} \\ F_{yx} & F_{yy} & F_{ya} \\ F_{ax} & F_{ay} & F_{aa} \end{vmatrix} \equiv 0.$$

Because of 6), the equations

$$F_{xx}\mu' + F_{xy}\nu' + F_{xa} = 0,$$
$$F_{yx}\mu' + F_{yy}\nu' + F_{ya} = 0,$$

determine μ' and ν' at any point along $x = \mu(a)$, $y = \nu(a)$, as

$$(A) \qquad \mu' = \frac{F_{xy}\,F_{ya} - F_{yy}\,F_{xa}}{F_{xx}\,F_{yy} - F_{xy}^2}, \qquad \nu' = \frac{F_{xy}\,F_{xa} - F_{xx}\,F_{ya}}{F_{xx}\,F_{yy} - F_{xy}^2}.$$

Since $\mu'(a)$ and $\nu'(a)$ are not both identically zero it follows that at least one of the inequalities $F_{ax}(x, y, a) \not\equiv 0$, $F_{ay}(x, y, a) \not\equiv 0$ must hold along $x = \mu(a)$, $y = \nu(a)$. This establishes the theorem.

Theorem II. *A sufficient condition that the given locus $x = \mu(a)$, $y = \nu(a)$, where $|\mu'(a)| + |\nu'(a)| \not\equiv 0$, be a nodal locus of the curves $F(x, y, a) = 0$ is that the relations*

$$1)\ F \equiv 0, \quad 2)\ F_x \equiv 0, \quad 3)\ F_y \equiv 0, \quad 4)\ F_{xx}F_{yy} - F_{xy}^2 \neq 0,$$

be satisfied along this locus.

Proof. The conditions of the theorem of §10 are fulfilled at every point of the given locus. Hence the theorem follows at once.

THEOREM III. *If, at* (x_0, y_0, a_0),

$$1)\ F_x = 0,\quad 2)\ F_y = 0,\quad 3)\ F_{xx}F_{yy} - F_{xy}^2 \neq 0,$$

then $$F_x(x, y, a) = 0,\quad F_y(x, y, a) = 0$$

define $$x = \mu(a),\quad y = \nu(a),$$

where $$a)\quad \mu(a),\ \nu(a)\ \textit{are analytic at } a_0,$$

$$b)\quad \mu(a_0) = x_0,\quad \nu(a_0) = y_0.$$

If, furthermore,

$$4)\ F(\mu, \nu, a) \equiv 0,\quad 5)\ F_{ax}(\mu, \nu, a) \not\equiv 0,\ \textit{or}\ F_{ay}(\mu, \nu, a) \not\equiv 0,$$

then $$c)\qquad |\mu'(a)| + |\nu'(a)| \not\equiv 0,$$

and $$x = \mu(a),\quad y = \nu(a)$$

define, in the neighborhood of (x_0, y_0, a_0), *a nodal locus of the curves* $F(x, y, a) = 0$.

Proof. Conditions 1), 2), 3) insure, by the ordinary Implicit Function Theorem, the existence of two functions $x = \mu(a)$, $y = \nu(a)$ satisfying conclusions a) and b), and such that

$$1')\ F_x(\mu, \nu, a) \equiv 0,\qquad 2')\ F_y(\mu, \nu, a) \equiv 0.$$

That conditions 3), 5) are sufficient that $\mu'(a)$, $\nu'(a)$ do not both vanish identically, follows at once from the equations (A) for $\mu'(a)$, $\nu'(a)$ obtained in the proof of Theorem I, above. Finally, 4), 1'), 2'), 3) constitute precisely the sufficient conditions of Theorem II above, which completes the proof of the Theorem.

THEOREM IV.* *A necessary and sufficient condition that a given nodal locus* $x = \mu(a)$, $y = \nu(a)$, *where* $|\mu'(a)| + |\nu'(a)| \not\equiv 0$, *of the curves* $F(x, y, a) = 0$ *be an envelope of one of the branches at the node, is that*

$$F_{aa}(\mu, \nu, a) \equiv 0.$$

* This theorem is due to M. J. M. Hill, *Proc. London Math. Soc.*, vol. 22 (1891), p. 222.

Proof. Necessary condition. Since $x = \mu(a)$, $y = \nu(a)$ is a nodal locus of $F(x, y, a) = 0$,

$$F_x(\mu, \nu, a) \equiv 0, \quad F_y(\mu, \nu, a) \equiv 0, \quad F_a(\mu, \nu, a) \equiv 0;$$

then

1) $$F_{xx}dx + F_{xy}dy + F_{xa}da \equiv 0,$$

2) $$F_{yx}dx + F_{yy}dy + F_{ya}da \equiv 0,$$

3) $$F_{ax}dx + F_{ay}dy + F_{aa}da \equiv 0,$$

are consistent along the given locus.

Since, by hypothesis, the slope of the nodal locus is equal to the slope of one of the branches at the node, dy/dx from the above set must satisfy the equation

4) $$F_{xx}(dx)^2 + 2F_{xy}\,dx \cdot dy + F_{yy}(dy)^2 = 0,$$

which determines the slopes of the two branches at the node. Multiply 1) by dx, 2) by dy, and add:

5) $$F_{xx}(dx)^2 + 2F_{xy}dx \cdot dy + F_{yy}(dy)^2 + F_{xa}dx \cdot da + F_{ya}dy \cdot da \equiv 0,$$

whence $$da(F_{xa}dx + F_{ya}dy) \equiv 0$$

along the locus. Since $da \neq 0$, it follows that

6) $$F_{xa}dx + F_{ya}dy \equiv 0$$

along the locus. Hence, from 3), it follows that $F_{aa}(\mu, \nu, a) \equiv 0$, as was to be proved.

Sufficient condition. Since $x = \mu(a)$, $y = \nu(a)$ is a nodal locus, 1), 2), 3), and therefore 5) are true. Since, by hypothesis, $F_{aa}(\mu, \nu, a) \equiv 0$, 3) insures that 6) is true; 5) then reduces to 4); that is, dy/dx from set 1), 2), 3) satisfies 4), and the locus is an envelope of a branch of $F(x, y, a) = 0$ through the node. This completes the proof of the theorem.

12. Conditions for the Real Case. If $F(x, y, a)$ is real when x, y, a are real, the theorems of §11 read for *a*) nodes, *b*) conjugate points, according as we change the condition $F_{xx}F_{yy} - F_{xy}^2 \neq 0$ to read

a) $F_{xx}F_{yy} - F_{xy}^2 < 0$, *b*) $F_{xx}F_{yy} - F_{xy}^2 > 0$, respectively.

Example 1. Consider the family of lemniscates:

$$F(x, y, a) \equiv \{(x+a)^2 + (y-a)^2\}^2 - 4b^2\{(x+a)^2 - (y-a)^2\} = 0.$$

The nodal locus is the straight line $x = -a$, $y = a$.

$$F_{aa}(x, y, a) \equiv 8\{(x+a)^2 + (y-a)^2 + (x-y+2a)^2\}.$$

Since $F_{aa}(\mu, \nu, a) \equiv 0$, the nodal locus in an envelope of one of the branches through the node.

Example 2. Consider the family of curves

$$F(x, y, a) \equiv x^3 + (y-a)^3 - 3bx(y-a) = 0.$$

The nodal locus is $x = 0$, $y = a$.

$$F_{aa}(x, y, a) \equiv 6(y-a), \quad F_{aa}(\mu, \nu, a) \equiv 0.$$

Hence the nodal locus is an envelope.

13. The Locus of Double Points with Coincident Tangents. As stated in §10, Theorem II, necessary and sufficient conditions that the point (x_0, y_0) be, on the curve $F(x, y) = 0$, a double point with coincident tangents are that

$$F = 0, \quad F_x = 0, \quad F_y = 0, \quad F_{xx}F_{yy} - F_{xy}^2 = 0,$$

at the point, while F_{xx}, F_{xy}, F_{yy} do not all vanish there.

THEOREM I. *Necessary conditions that a given locus $x = \mu(a)$, $y = \nu(a)$, where $|\mu'(a)| + |\nu'(a)| \neq 0$, be a locus of double points with coincident tangents for the curves $F(x, y, a) = 0$ are:*

$$1)\ F \equiv 0, \qquad 2)\ F_x \equiv 0, \qquad 3)\ F_y \equiv 0, \qquad 4)\ F_a \equiv 0,$$

$$5)\ F_{xx} : F_{xy} : F_{xa} \equiv F_{yx} : F_{yy} : F_{ya} \equiv F_{ax} : F_{ay} : F_{aa},$$

along the locus, while

$$6) \qquad F_{xx},\ F_{xy},\ F_{yy}$$

do not all vanish simultaneously at any point of the given locus.

Proof. Conditions 1), 2), 3), 6), and

$$7) \qquad F_{xx}F_{yy} - F_{xy}^2 \equiv 0$$

follow at once from the fact that the above requirements for a double point with coincident tangents hold at every point of the given locus. Differentiating 1), we obtain condition 4) by using 2) and 3). From the identities 2), 3), 4), we obtain the consistent equations

$$8)\qquad F_{xx}\mu' + F_{xy}\nu' + F_{xa} = 0,$$

$$9)\qquad F_{yx}\mu' + F_{yy}\nu' + F_{ya} = 0,$$

$$10)\qquad F_{ax}\mu' + F_{ay}\nu' + F_{aa} = 0.$$

From 7), and the consistency of 8) and 9), it follows that

$$11)\qquad F_{xx} : F_{xy} : F_{xa} \equiv F_{yx} : F_{yy} : F_{ya}.$$

From $F_{xx} : F_{xy} \equiv F_{xa} : F_{ya}$, and the consistency of 8) and 10), it follows that

$$12)\qquad F_{xx} : F_{xy} : F_{xa} \equiv F_{ax} : F_{ay} : F_{aa}.$$

Condition 5) is obtained by combining 11) and 12). This completes the proof of the theorem.

THEOREM II. *Sufficient conditions that a given locus $x = \mu(a)$, $y = \nu(a)$, where $|\mu'(a)| + |\nu'(a)| \neq 0$, be a locus of double points with coincident tangents for the curves $F(x, y, a) = 0$, are:*

$$1)\ F \equiv 0,\quad 2)\ F_x \equiv 0,\quad 3)\ F_y \equiv 0,\quad 4)\ F_{xx}F_{yy} - F_{xy}^2 \equiv 0,$$

along the locus, while

$$5)\qquad F_{xx},\ F_{xy},\ F_{yy}$$

do not all vanish simultaneously at any point of the given locus.

Proof. The conditions for such a point are fulfilled at every point of the given locus, and the thorem is established.

THEOREM III. *A necessary and sufficient condition that a given locus of double points with coincident tangents,*

$$\mu(a_0) = x_0\,,\qquad \nu(a_0) = y_0,\qquad |\mu'(a)| + |\nu'(a)| \neq 0,$$

of the curves $F(x, y, a) = 0$, be an envelope of these curves is that

$$F_{aa}(\mu, \nu, a) \equiv 0.$$

Proof. Since $x = \mu(a)$, $y = \nu(a)$ is a locus of double points with coincident tangents, we have along it

$$1)\ F \equiv 0, \qquad 2)\ F_x \equiv 0, \qquad 3)\ F_y \equiv 0, \qquad 4)\ F_a \equiv 0,$$

$$5)\ F_{xx} : F_{xy} : F_{xa} \equiv F_{yx} : F_{yy} : F_{ya} \equiv F_{ax} : F_{ay} : F_{aa},$$

while

$$6) \qquad F_{xx},\ F_{xy},\ F_{yy}$$

do not all vanish at any point of the given locus.

Necessary condition. Since the slope of the given locus is, by hypothesis, equal at every point to the slope of the double point tangent at that point, we have, using **5**) :

$$\begin{aligned} \nu'(a) : \mu'(a) &\equiv - F_{xx}(\mu, \nu, a) : F_{xy}(\mu, \nu, a) \\ &\equiv - F_{yx}(\mu, \nu, a) : F_{yy}(\mu, \nu, a) \\ &\equiv - F_{ax}(\mu, \nu, a) : F_{ay}(\mu, \nu, a) ; \end{aligned}$$

hence

$$7) \qquad F_{xx}\mu' + F_{xy}\nu' \equiv 0,$$

$$8) \qquad F_{yx}\mu' + F_{yy}\nu' \equiv 0,$$

$$9) \qquad F_{ax}\mu' + F_{ay}\nu' \equiv 0,$$

are necessary conditions along the locus.

From these and the identities 8), 9), 10) under Theorem I, it follows that

$$F_{ax}(\mu, \nu, a) \equiv 0, \quad F_{ay}(\mu, \nu, a) \equiv 0, \quad F_{aa}(\mu, \nu, a) \equiv 0.$$

This establishes the necessary condition, and incidentally shows that $F_{ax}(\mu, \nu, a) \equiv 0$ and $F_{ay}(\mu, \nu, a) \equiv 0$ are equivalent necessary conditions.

Sufficient conditions. Let $F_{aa}(\mu, \nu, a) \equiv 0$; then, from 5)

$$F_{ax}(\mu, \nu, a) \equiv 0, \quad F_{ay}(\mu, \nu, a) \equiv 0.$$

The equations 7), 8), 9) are then true, and, because of 5),

$$\begin{aligned} \nu'(a) : \mu'(a) &\equiv - F_{xx}(\mu, \nu, a) : F_{xy}(\mu, \nu, a) \\ &\equiv - F_{yx}(\mu, \nu, a) : F_{yy}(\mu, \nu, a), \end{aligned}$$

that is, the slope of the given locus is equal to that of the double point tangent, and the locus is an envelope. This completes the proof of the theorem.

14. Examples. *Example* 1. Consider the curves

$$F(x, y, a) \equiv (x-y)^2 - (x+y-2a)^3 = 0.$$

The cusp locus is $x = a$, $y = a$.

$$F_{aa}(x, y, a) \equiv -24(x+y-2a).$$

Since $F_{aa}(a, a, a) \equiv 0$, the cusp locus is an envelope.

Example 2. Consider the family of cissoids

$$F(x, y, a) \equiv x^3 + (x-1)(y-a)^2 = 0.$$

The cusp locus is $x = 0$, $y = a$.

Since $F_{aa}(x, y, a) \equiv 2(x-1) \neq 0$ along the cusp locus, this locus is not an envelope.

Example 3. Consider the curves

$$F(x, y, a) \equiv y^2 - 2y(x-a)^2 + (x-a)^4 - (x-a)^5 = 0.$$

The cusp lo us is $x = a, y = 0$. These cusps are of the second kind, both branches of the curve lying on the same side of the cusp tangent.

$$F_{aa}(x, y, a) \equiv -4y + 12(x-a)^2 - 20(x-a)^3.$$

Since $F_{aa}(\mu, \nu, a) \equiv 0$, the cusp locus is an envelope.

Example 4. Consider the curves

$$F(x, y, a) \equiv (x+y)^2 - (x-y-2a)^4 + (x-y-2a)^6 = 0.$$

The line $x = a$, $y = -a$, is the locus of tacnodes of the curves. Since $F_{aa}(x, y, a) \equiv -48(x-y-2a)^2 + 120(x-y-2a)^4 \equiv 0$ along this locus of tacnodes, the locus is an envelope of the curves.

Example 5. Consider the curves

$$F(x, y, a) \equiv (y-a)^2 + (x-a)^4 + (y-a)^4 = 0.$$

Each curve a of this family has but one real point, $x = a$, $y = a$;

$$F_{xx}(x, y, a) \equiv 12(x-a)^2,\ F_{xy}(x, y, a) \equiv 0,\ F_{yy}(x, y, a) \equiv 2 + 12(y-a)^2.$$

The conditions for double points are satisfied at every point of $x = a$, $y = a$, which is therefore the locus of isolated double points of the given curves. The equation for the slopes of the branches through such a point,

$$(x - x_0)^2 F_{xx} + 2(x - x_0)(y - y_0)\, F_{xy} + (y - y_0)^2\, F_{yy} = 0,$$

shows that the slopes of the two branches through the point (a_0, a_0) are both zero: we shall call this the slope of the coincident tangents to the curves at the isolated double points.

$$F_{\alpha\alpha}(x, y, a) \equiv 2 + 12(x - a)^2 + 12(y - a)^2.$$

Since $F_{\alpha\alpha}(\mu, \nu, a) \equiv 2 \neq 0$, the locus of isolated points is not an envelope.

The family

$$F(x, y, a) \equiv y^2 + (x - a)^4 + y^4 = 0$$

has $x = a$, $y = 0$ as a locus of isolated double points with coincident slopes. Since $F_{\alpha\alpha}(a, 0, a) \equiv 0$, this locus is an envelope.

ON THE SOLUTIONS OF ORDINARY LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS OF THE THIRD ORDER

By George D. Birkhoff

Introduction. Given a linear differential equation of the third order

$$y''' + py'' + qy' + ry = 0, \qquad (a \leqq x \leqq b), \tag{1}$$

in which the coefficients p, q, r are real functions of x, continuous together with their derivatives of all orders, but otherwise unrestricted; what is the general character of the solutions and how do they depend on these coefficients? This rather interesting question lies almost wholly untreated, and it is the aim of the present paper to consider it.

The analogous question in the case of an equation of the first order

$$y' + py = 0$$

is at once answered. By means of the explicit form of the general solution

$$y = c\, e^{-\int p\,dx}, \qquad (c, \text{ a constant}),$$

we see first that *y cannot vanish*, unless it does so identically. Also any function y_1 which does not vanish is the solution of such an equation, namely

$$y' - \frac{y_1'}{y_1} y = 0.$$

Thus the general characterization of the solutions has been obtained. The exact nature of the dependence of the solutions upon the coefficient p is exhibited by the explicit formula.

If the equation be of the second order,

$$y'' + py' + qy = 0,$$

the general solution has the form

$$c_1 y_1 + c_2 y_2, \qquad (c_1, c_2, \text{ constants}),$$

where y_1 and y_2 are any pair of linearly independent particular solutions. It

is an immediate consequence of the fundamental existence theorem for ordinary differential equations that

$$W = y_1 y_2' - y_2 y_1'$$

cannot vanish. For suppose that this expression did vanish for $x = \xi$. In this event we could determine α and β, not both zero, so that the two equations

$$\alpha y_1(\xi) + \beta y_2(\xi) = 0, \qquad \alpha y_1'(\xi) + \beta y_2'(\xi) = 0$$

would hold. The solution

$$y = \alpha y_1 + \beta y_2$$

and its first derivative would then vanish at $x = \xi$. But, by the existence theorem referred to, there is but one such solution, namely $y = 0$. Accordingly y_1 and y_2 would be linearly dependent, contrary to our hypothesis. Thus W cannot vanish. Furthermore if y_1 and y_2 are any functions for which $W \neq 0$, these functions will form a pair of linearly independent solutions of the linear differential equation of the second order

$$\begin{vmatrix} y'' & y' & y \\ y_1'' & y_1' & y_1 \\ y_2'' & y_2' & y_2 \end{vmatrix} = 0,$$

the coefficient of y'' being precisely $-W$. This coefficient can be made unity by division of both members of the equation by $-W$.

The analytic condition $W \neq 0$, which has been shown to characterize y_1 and y_2, tells us that the ratio of y_1 to y_2 increases or decreases throughout the interval considered. Let now y_1 and y_2 be regarded as the coördinates of a point in the projective line. As x increases through the interval, the point will describe the line in one sense continually. The elementary *Separation Theorem* concerning the roots of $y_1 = 0$ and $y_2 = 0$ is an evident consequence of this fact: *the roots of $y_1 = 0$ and $y_2 = 0$ will separate each other*. The above results give a characterization of the solutions, and thus answer the first part of our question for a linear differential equation of the second order.

As the most important of the theorems which deal with the dependence of solutions upon the coefficients we cite the following *Comparison Theorem*: suppose two equations

$$y'' + q_1 y = 0 \qquad \text{and} \qquad y'' + q_2 y = 0, \qquad q_1 > q_2,$$

to be given; then there will be at least one zero of any solution of the first equation between two zeros of any solution of the second, or less precisely, *an increase of q in the equation*

$$y'' + qy = 0$$

*brings the zeros of the solutions closer together.**

Very analogous theorems exist in the case of the equation of the third order, and it is with these that we are concerned here.

1. Geometrical Interpretation. If one writes

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix},$$

a necessary and sufficient condition that y_1, y_2, y_3 form three linearly independent solutions of some linear differential equation of the third order (1) is seen to be $W \neq 0$, just as the analogous fact was seen in the case of the second order equation. An explicit formula for W is obtained by noting that one has

$$\frac{dW}{dx} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} = -pW,$$

by (1), so that W is given by the formula

$$W = c\, e^{-\int p dx}, \qquad (c \neq 0). \tag{2}$$

*Take for example the equations $y'' + \lambda_1 y = 0$ and $y'' + \lambda_2 y = 0$, where λ_1 and λ_2 are positive constants, λ_1 greater than λ_2. The interval between zeros is fixed in both cases, $\pi/\sqrt{\lambda_1}$ in the first and $\pi/\sqrt{\lambda_2}$ in the second case.

The theorem is more general than at first might appear since any equation of the second order may be given the normal form,

$$y'' + qy = 0,$$

by a simple transformation.

The reader is referred for the proof of this and similar results to articles by Professor Bôcher in the *Bulletin of the American Mathematical Society*. vol. 4, pp. 295–313, pp. 365–376, 1897–1898, as well as to the original paper of Sturm, *Journal de Mathématique*, vol. 1, pp. 106–186, 1836.

To a set of linearly independent solutions y_1, y_2, y_3 of a given equation (1), whose general solution is then

$$c_1 y_1 + c_2 y_2 + c_3 y_3,$$

there will correspond an *integral curve* C defined by the equations

$$w_1 = y_1(x), \qquad w_2 = y_2(x), \qquad w_3 = y_3(x),$$

in which w_1, w_2, w_3 are the homogeneous coördinates of a point in the projective plane referred to a certain triangle of reference. It is clear that any other integral curve D obtained by making a different choice of the three linearly independent solutions will be projectively equivalent to C, since the new solutions can be expressed linearly in terms of y_1, y_2, y_3.

The equation of a tangent line to C is

$$\begin{vmatrix} w_1 & w_2 & w_3 \\ y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \end{vmatrix} = 0.$$

The homogeneous coördinates of this line are therefore

$$(3) \qquad z_1 = \frac{y_2 y_3' - y_3 y_2'}{W}, \quad z_2 = \frac{y_3 y_1' - y_1 y_3'}{W}, \quad z_3 = \frac{y_1 y_2' - y_2 y_1'}{W},$$

the divisor W being so chosen that z_1, z_2, z_3 are solutions of the *adjoint equation* to (1):

$$(4) \qquad y''' - (py)'' + (qy)' - ry = 0.$$

The fact that z_1, z_2, z_3 are solutions of (4) may be verified by substitution. The relation between (1) and (4) is entirely reciprocal in its nature, for equation (1) is also the adjoint to (4). Furthermore, the functions z_1, z_2, z_3 are linearly independent solutions of (4) since otherwise the curve C would reduce to a point or straight line, and y_1, y_2, y_3 would be also linearly dependent. The reader is referred to Darboux, *Théorie des Surfaces*, vol. II, book IV, chap. 5 for a discussion of the properties of the adjoint equation.

The identities

$$(5) \qquad \begin{aligned} y_1z_1 + y_2z_2 + y_3z_3 &= 0, \\ y_1'z_1 + y_2'z_2 + y_3'z_3 &= 0, \\ y_1''z_1 + y_2''z_2 + y_3''z_3 &= 1, \end{aligned}$$

are obvious on substitution of the expressions given in (3) for z_1, z_2, z_3. These equations serve also to define z_1, z_2, z_3 in terms of y_1, y_2, y_3.

Returning now to the curve C, we see that it is a *continuous* curve in the projective plane (i.e. if the projective plane be mapped on the ordinary Euclidean plane so that P_0 is a point of C in the finite plane, then C is continuous at P_0 in the ordinary sense) since y_1, y_2, y_3 are continuous functions, not all simultaneously zero ($W \neq 0$). Likewise C has a *continuously turning tangent* in the projective plane (i.e. if the projective plane be mapped on the ordinary Euclidean plane so that P_0 is a point of C in the finite plane, then the direction of the tangent line varies continuously along the curve at P_0), since z_1, z_2, z_3 are also continuous functions, not all simultaneously zero. Thus C is a *regular curve* in the projective plane. The fact that y_1, y_2, y_3 have derivatives in x of all orders may be taken account of by saying that C is *completely regular*.

The significance of the condition $W \neq 0$ is that there are no points of inflection. In fact, by definition, the tangent at such a point has contact with the curve of higher order than the first, and hence if

$$a_1w_1 + a_2w_2 + a_3w_3 = 0$$

is the tangent line, one has simultaneously

$$\begin{aligned} a_1y_1 + a_2y_2 + a_3y_3 &= 0, \\ a_1y_1' + a_2y_2' + a_3y_3' &= 0, \\ a_1y_1'' + a_2y_2'' + a_3y_3'' &= 0. \end{aligned}$$

This is possible if and only if W vanishes. *The curve C may therefore be any completely regular curve without any point of inflection.* The reader will remember that the only condition on y_1, y_2, y_3 that they form linearly independent solutions of *some* equation (1) is the condition $W \neq 0$.

If we consider the projective plane as lying in the ordinary Euclidean plane, the curve C possesses a tangent which rotates continuously in one

direction until the curve passes through the 'line at infinity' which is the image of any arbitrary line in the projective plane. It then returns with a tangent rotating in the reverse direction (consider for example a conic, which has no point of inflection).

A transformation

$$\bar{y} = \lambda y, \qquad (\lambda \neq 0),$$

followed by a division by λ, leaves (1) linear and of course does not change the curve C. By choosing

$$\lambda = e^{-\frac{1}{3}\int p\,dx},$$

the coefficient of the second term of (1) will reduce to zero and that equation will take on the simple form

$$y''' + qy' + ry = 0, \tag{6}$$

to which the adjoint equation is

$$y''' + qy' + (q' - r)y = 0. \tag{7}$$

It may be observed in passing that this last equation might also have been obtained from (4) by a like transformation in which however

$$\lambda = e^{\frac{1}{3}\int p\,dx}.$$

A simplification introduced by giving our equations this normal form is that W reduces by (2) to a non-vanishing constant.

From this point forth we shall often employ our equations in this simplified form. The curve C may then be looked upon as given either in normalized point coördinates by (6) or in normalized line coördinates by (7).

The above interpretation of the solutions of an ordinary linear homogeneous differential equation by means of a curve is a well-known one. The second and third chapters of Professor Wilczynski's *Projective Differential Geometry* contain a full development of it.

2. General Separation Theorem. The above geometrical representation furnishes us at once with the negative conclusion—not to be seen very readily without use of this representation perhaps—that it is possible to choose (a) the differential equation (1), and (b) three linearly independent solutions y_1, y_2, y_3, of the same, so that the zeros of y_1, y_2, y_3 will succeed each

other in any arbitrarily prescribed order. It will suffice to show that there exists a completely regular curve C without points of inflection which cuts the sides of the triangle of reference in any arbitrarily prescribed succession, since such a curve is the integral curve of some equation (1). The existence of such a curve may be readily seen as follows: Let C be drawn up to a point P not on a side of the triangle of reference; the curve C may be extended beyond P without the introduction of points of inflection, so as to cut any assigned side of the triangle. The method of extension is given in the adjoining figure

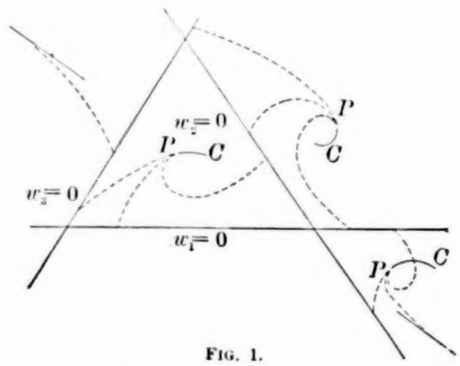

FIG. 1.

(Fig. 1). By a succession of such extensions, C may be made to cut $w_1 = 0$, $w_2 = 0$, $w_3 = 0$ in the desired order.

It is only when we consider the solutions y_1, y_2, y_3 and z_1, z_2, z_3 together that we obtain a Separation Theorem concerning the order of the zeros of these six functions.

GENERAL SEPARATION THEOREM. *Between any two successive zeros of* y_i *(or of* z_i*) there are an odd number of zeros of* y_k *and* z_l *together, where* (i, k, l) *is any permutation of* $(1, 2, 3)$.

Proof. If we can prove that between any pair of successive zeros of y_1 there are an odd number of zeros of y_2 and z_3 together, the theorem will follow by considerations of symmetry and duality. We may suppose without loss of generality that the line $w_1 = 0$ of the projective plane goes over into the line at infinity in the Euclidean plane. The arc of C corresponding to values of x between the two values for which $y_1 = 0$ is then a single *branch* of the curve. The line $w_2 = 0$ is some line in the finite plane. The geometrical interpretation of the condition $y_2 = 0$ is then that the curve C cuts this fixed line $w_2 = 0$; the interpretation of the condition $z_3 = 0$ is that the tangent line is parallel to

the line $w_2 = 0$, in other words, passes through the intersection of $w_1 = 0$ and $w_2 = 0$.

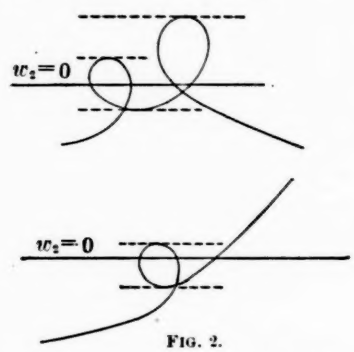

FIG. 2.

Now (Fig. 2) if the arc enters on one side of $w_2 = 0$ in this plane and leaves on the other there are an odd number of zeros of y_2 and an even number of zeros of z_3; otherwise there are an even number of zeros of y_2, and an odd number of zeros of z_3. In either case the theorem holds.

Self-Adjoint Equations. If the integral curve C is self-dual, equations (6) and (7) are identical since the normal form (6) is unique. The condition for this is plainly

$$r = \frac{q'}{2}.$$

In this case, which possesses especial interest, z_1, z_2, z_3 must be linearly expressible in terms of y_1, y_2, y_3:

$$z_1 = c_{11}y_1 + c_{12}y_2 + c_{13}y_3,$$

$$z_2 = c_{21}y_1 + c_{22}y_2 + c_{23}y_3,$$

$$z_3 = c_{31}y_1 + c_{32}y_2 + c_{33}y_3.$$

Since the solutions z_1, z_2, z_3 are linearly independent, the determinant of the coefficients is not zero. Hence one finds from the first equation (5), by substitution of the expressions above for z_1, z_2, z_3, that

$$c_{11}y_1^2 + c_{22}y_2^2 + c_{33}y_3^2 + (c_{23} + c_{32})y_2y_3 + (c_{31} + c_{13})y_3y_1 + (c_{12} + c_{21})y_1y_2 = 0,$$

the equation of a conic unless

$$c_{11} = c_{22} = c_{33} = 0, \qquad c_{23} = -c_{32}, \qquad c_{31} = -c_{13}, \qquad c_{12} = -c_{21}.$$

These conditions cannot all hold since, if they do, the determinant of the coefficients will be zero. *The integral curve C must be an arc of the conic, or*

the entire conic. In the latter case as x increases from a to b, the corresponding point P may traverse the conic in one sense any number of times, making q complete circuits and a partial circuit.

The above considerations lead us at once to the following theorem:

SEPARATION THEOREM. *Self-Adjoint Case. There exist solutions of any equation* (1) *reducible to self-adjoint form which nowhere vanish in* (a, b). *If* y_1 *and* y_2 *are solutions of the equation with at least one zero in* (a, b), *the zeros of* y_1 *and* y_2 *either separate each other singly or in pairs.*

Proof. In fact, there exist straight lines which do not cut the conic, and therefore solutions which do not vanish, since the condition

$$a_1y_1 + a_2y_2 + a_3y_3 \neq 0$$

is equivalent to the condition that the conic does not cut the line

$$a_1w_1 + a_2w_2 + a_3w_3 = 0.$$

Also, of two straight lines which do cut the conic, say in M_1, M_2 and N_1, N_2, respectively, either M_1 and M_2 separate N_1 and N_2 on the conic, in which case the zeros of the corresponding solutions occur alternately as we traverse the conic; or these pairs do not separate each other, in which case we have first two zeros of the one solution, then two of the other, and so on.

If C is merely an arc of the conic no solution vanishes more than twice in (a, b) although the theorem still holds.

Transformation of the Independent Variable. In obtaining the normal form (6) only a multiplicative transformation of the dependent variable,

$$\bar{y} = \lambda y, \qquad (\lambda \neq 0),$$

was made use of, while it is known that (1) also remains linear when any transformation of the independent variable

$$\bar{x} = \phi(x), \qquad (\phi' \neq 0),$$

is made. From the standpoint of the geometrical interpretation, the second transformation changes the parameter of the curve.

What is the simplest form of the equation obtainable by a suitable combination of both transformations?

First, if the equation be reducible to self-adjoint form the locus of the

point (y_1, y_2, y_3) is a conic, and a suitable transformation of the variables will reduce y_1, y_2, y_3 to the form

$$\bar{y}_1 = \cos \bar{x}, \qquad \bar{y}_2 = \sin \bar{x}, \qquad \bar{y}_3 = 1,$$

no matter how many times the integral conic be traversed. Hence a normal form in this case is

$$\bar{y}''' + \bar{y}' = 0, \qquad (\bar{a} \leqq \bar{x} \leqq \bar{b}).$$

Likewise, since the equation (6) may be written

$$y''' + qy' + \left(\frac{q'}{2} + R\right)y = 0,$$

one sees that the transformation which reduces the self-adjoint equation

$$(8) \qquad y''' + qy' + \frac{q'}{2}\, y = 0$$

to the preceding normal form will at the same time reduce (6) to an equation

$$(9) \qquad \bar{y}''' + \bar{y}' + \bar{R}\, \bar{y} = 0, \qquad (\bar{a} \leqq \bar{x} \leqq \bar{b}).$$

The Forsyth-Laguerre normal form is

$$\bar{y}''' + \bar{R}\, \bar{y} = 0;$$

in this case the self-adjoint equation (8) has been transformed to the equation

$$\bar{y}''' = 0$$

with solutions $1, \bar{x}, \bar{x}^2$. Since the equations

$$w_1 = 1, \qquad w_2 = \bar{x}, \qquad w_3 = \bar{x}^2$$

give us each point of the conic once and only once, this transformation is not proper unless the interval (a, b) is so small that the integral curve of (8) is an arc of the conic.

3. Separation Theorem for Regular Intervals. The somewhat indefinite character of the first Separation Theorem is due in large measure to the fact that the integral curve C may have a very complicated form; in par-

ticular it may have double points and double tangents. A *regular interval* (a, b) is defined to be an interval such that the corresponding integral curve C is either an oval, or has no double points or double tangents except where C touches but does not intersect itself. On account of the fact that double points and double tangents are dual notions the interval (a, b) will be regular for the adjoint equation (4) if it is regular for (1) itself, and conversely. It is obvious that any interval can be separated into a finite number of regular intervals.

SEPARATION THEOREM FOR REGULAR INTERVALS. *If (a, b) is a regular interval there will exist a solution y_1 of (1) which is of one sign in (a, b) and a family of solutions $c_2 y_2 + c_3 y_3$ such that the zeros of any two members of the family separate each other.*

Proof. The theorem is clearly true if the integral curve is an oval, so that this possibility is at once disposed of.

Assume first that the arc AB corresponding to the interval (a, b) does not cut some line of the projective plane, which is taken to be the line at infinity in the Euclidean plane. On account of the hypothesis that (a, b) is regular, it is clear that AB will be *spiraliform* (Fig. 3). We are not excluding the possibility that A or B or both lie on the line at infinity.

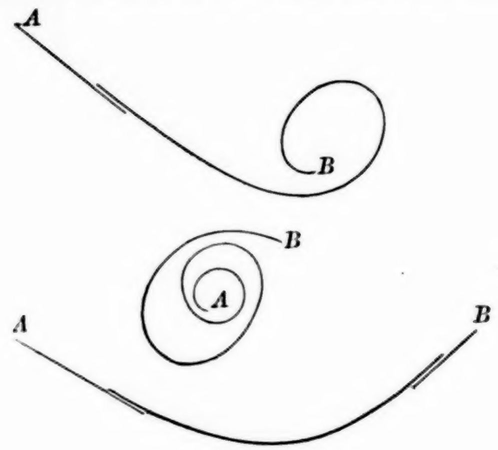

FIG. 3.

We shall now show that there exists such a line in every case. Construct the tangent l at a point P of AB. As P moves from A to B along the arc AB, the line l will not at first cut AP. There must, however, be a least value

of x, say ξ, such that for slightly greater values the corresponding tangent l will cut AP; otherwise the tangent line at B does not cut AB and we are brought back to the previous case. Furthermore, the tangent corresponding to $x = \xi$ obviously passes through A. If this particular tangent line be chosen as the line at infinity, it is clear that AP will correspond to a single branch of AB, without inflection point or double point, of course, and that accordingly there exist lines l, which do not touch AP. Now take such a line l, as the line at infinity, and the arc AP will have the form indicated in Figure 4.

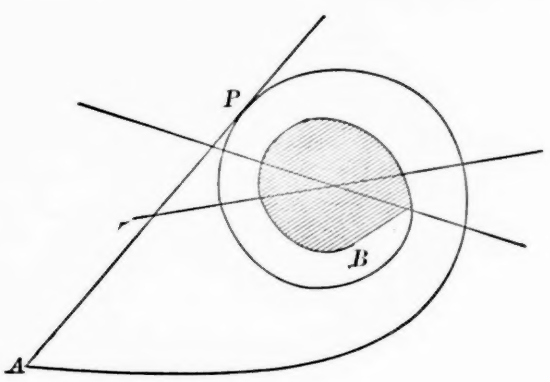

FIG. 4.

From this figure it is clear that, since there are no double points of intersection, the entire arc AB must lie between the arc AP and the tangent at P. Hence this case is reduced to that first considered.

Thus the integral curve C is spiraliform in all cases. It should be noted that an oval is to be looked upon as a limiting form of a spiral.

Any line which does not have a point in common with AB will correspond to a solution y_1, which does not vanish for $a \leqq x \leqq b$. Likewise all the lines through a point in the interior of the integral curve (see shaded region in Fig. 4) will correspond to a linear family

$$c_2y_2 + c_3y_3$$

whose zeros separate each other.

4. An Auxiliary Formula. We shall now derive an auxiliary formula upon which for the most part the further theorems will be based.

Let y_1, y_2, y_3 be three linearly independent solutions of a given linear

differential equation (1), and z_1, z_2, z_3 the corresponding solutions of the adjoint equation (4), and let us write

$$\phi(x, \xi) = y_1(x)z_1(\xi) + y_2(x)z_2(\xi) + y_3(x)z_3(\xi). \tag{10}$$

The relations (5) are then

$$\phi(\xi, \xi) = \phi_1(\xi, \xi) = 0, \qquad \phi_2(\xi, \xi) = 1, \tag{11}$$

where our notation is

$$\phi_1(x, \xi) = \frac{\partial\phi}{\partial x}, \qquad \phi_2(x, \xi) = \frac{\partial^2\phi}{\partial x^2}, \text{ etc.}$$

Take now (1) in the special form (6) and consider the modified equation

$$y''' + qy' + (r + R)y = 0, \tag{12}$$

which may be written in the form of a non-homogeneous linear differential equation

$$y''' + qy' + ry = f = -Ry. \tag{13}$$

The general solution of this equation may be written

$$y = c_1y_1 + c_2y_2 + c_3y_3 + \int_a^x \phi(x, \xi)\, f(\xi)d\xi. \tag{14}$$

In fact by direct differentiation and reduction by means of (11) one finds

$$y' = c_1y_1' + c_2y_2' + c_3y_3' + \int_a^x \phi_1(x, \xi)f(\xi)d\xi,$$

$$y'' = c_1y_1'' + c_2y_2'' + c_3y_3'' + \int_a^x \phi_2(x, \xi)f(\xi)d\xi,$$

$$y''' = c_1y_1''' + c_2y_2''' + c_3y_3''' + f(x) + \int_a^x \phi_3(x, \xi)\, f(\xi)d\xi.$$

Substitute these expressions for y and its derivatives in (13). This equation will then reduce to an identity by virtue of the fact that y_1, y_2, y_3 and $\phi(x, \xi)$ are solutions of (6), which proves that the expression for y given by (14) is the general solution of (13).

The function

$$\eta = c_1 y_1 + c_2 y_2 + c_3 y_3$$

is a solution of (6), and from the above equations we see that it satisfies the conditions

$$(15) \qquad y(a) = \eta(a), \qquad y'(a) = \eta'(a), \qquad y''(a) = \eta''(a).$$

Thus we may rewrite (14) in the final form

$$(16) \qquad y = \eta - \int_a^x \phi(x, \xi)\, R(\xi)\, y(\xi)\, d\xi,$$

in which y and η are corresponding solutions of (12) and (6) respectively, related by equations (15). This is the desired formula, and it represents, of course, nothing more than the solution of a special non-homogeneous linear differential equation.

5. Test for Regular Intervals. A regular interval (a, b) is said to be of the *first kind* if the tangent at A does not meet the arc AB at any second point (see Fig. 4), and of the *second kind* if the tangent at B does not meet the arc AB at any second point. According to this definition a regular interval may be of both kinds. We shall take the interval (a, b) to be of the first kind, stating, however, the results for both cases.

It follows at once from the definition, as has been noted, that if (a, b) is a regular interval for equation (1), it is also a regular interval for equation (4), the three solutions z_1, z_2, z_3 of (4), given by (3), now being interpreted as the coördinates of a point. Let the corresponding point locus be the curve D. Then D is obtained from C by a polar reciprocation and a suitable projection. Furthermore, if (a, b) is regular and of the *first kind* for (1), no line through B is tangent to the curve C at any other point than B (see Fig. 4). Therefore, in the curve D, no point on the tangent through B lies on the curve D, i. e. D is of the *second kind*.

A necessary condition that (a, b) is a regular interval is that *all the points of intersection of the tangent l at any point P of AB lie on one and the same side of P.* Thus in the preceding figure (Fig. 4) all the points of intersection of l lie in AP and none in PB. To prove the condition sufficient we have to show that, if it is satisfied, the integral curve AB can have no double points or double tangents except where the curve C touches but does not inter-

sect itself. Suppose there were such a double point $M = N$ where N lies in MB. Then the tangent at M cuts the curve at N, and the tangent at N cuts the curve at M. This is impossible if the condition is satisfied. If there were such a double tangent at M and N, where N lies in MB, the tangent line in the vicinity of M would cut the integral curve in the vicinity of N, and likewise the tangent line in the vicinity of N would cut the integral curve in the vicinity of M. This is also impossible if the condition is satisfied. Thus the validity of the condition has been demonstrated. Furthermore it is clear from the figure that (a, b) will be regular and *of the first kind* if no point of intersection of the tangent l lies in PB.

The analytic phrasing of the geometric condition can easily be given in terms of the function $\phi(x, \xi)$ defined in (10). For from the equations (11) it follows that the solution of (1) which corresponds to the tangent having a point of contact whose x is ξ, is $\phi(x, \xi)$. Accordingly $\phi(x, \xi)$ will not change sign for $x > \xi$, since the tangent at ξ does not cut the integral curve for points whose x is greater than ξ. Moreover, $\phi(x, \xi)$, regarded as a function of x, has a minimum at the point $x = \xi$, by (11). Therefore the analytic form of the condition is the following: *the interval (a, b) is regular and of the first (second) kind provided that $\phi(x, \xi)$ is positive or zero for $x > \xi\ (x < \xi)$.*

We are now in a position to develop the following test: *If (a, b) is a regular interval of the first (second) kind for the equation (6) it is also a regular interval of the same kind for the modified equation (12), provided that the inequality $R \leqq 0\ (R \geqq 0)$ obtains.*

Proof. The proof can immediately be made by means of the auxiliary formula (16) in which we take

$$\eta = \phi(x, a),$$

where a remains to be specified. This choice of η is possible since $\phi(x, a)$ is a solution of (6). We obtain then

$$y = \phi(x, a) - \int_a^x \phi(x, \xi)\, R(\xi)\, y(\xi) d\xi, \tag{17}$$

when y is that solution of the modified equation (12) which by (15) satisfies the conditions

$$y(a) = \phi(a, a) = 0, \quad y'(a) = \phi_1(a, a) = 0, \quad y''(a) = \phi_2(a, a) = 1. \tag{18}$$

Accordingly y is precisely the solution of the modified equation (12) which plays the same rôle for (12) that $\phi(x, a)$ does for the equation (6). Therefore if (a, b) is not regular and of the first kind for the modified equation there exists some a such that the corresponding y changes sign for $x > a$. Let x_0 be the least value of x for which this is true. Then we have the equation

$$0 = \phi(x_0, a) - \int_a^{x_0} \phi(x_0, \xi)\, R(\xi)\, y(\xi)\, d\xi.$$

But by hypothesis (a, b) is regular and of the first kind so that $\phi(x_0, a)$ and $\phi(x_0, \xi)\,(x_0 > \xi)$ are positive or zero. Moreover $R(\xi)$ is negative by hypothesis, and $y(\xi)$ is positive for ξ between a and x_0 by (18). Thus we have the sum of two positive or zero quantities equal to zero. Of these two quantities the second given by the definite integral must be negative; for the first factor $\phi(x_0, \xi)$ of the integrand can only be zero for isolated values of x since $\phi(x_0, \xi)$ is a not identically zero solution in ξ of the adjoint equation to (6), given by

$$y_1(x_0)z_1(\xi) + y_2(x_0)z_2(\xi) + y_3(x_0)z_3(\xi);$$

likewise the third factor of the integral cannot vanish except at x_0 and ξ on account of the way in which x_0 was chosen; finally the remaining factor $R(\xi)$ may be taken to be not identically zero. Hence the above equation is impossible.

The following special case of the test is important: *The interval (a, b) is a regular interval of the first (second) kind for an equation*

$$y''' + qy' + \left(\frac{q'}{2} + R\right) y = 0 \tag{19}$$

provided that $R \leqq 0\,(R \geqq 0)$ in the interval (a, b).

In this case it is only necessary to identify the equation (6) with the self-adjoint equation:

$$y''' + qy' + \frac{q'}{2}\, y = 0,$$

for which any interval (a, b) is regular and of both kinds, and to state the test for this special case.

Since any equation (6) can be written in the form (19) we have a method of separating any interval into a sequence of regular intervals the first of one kind, the next of the other, and so on, the end points of these intervals being given by those roots of $R = 0$ at which R changes sign.

The regular intervals thus chosen have a special geometric property which however I shall not stop to prove: *The successive osculating conics to the integral curve lie one within the other at all points of such an interval, and at the end points the curve C and the osculating conic hyperosculate.*

6. Intervals of Oscillation. Comparison Theorem. It remains to develop the nature of the dependence of solution and coefficients. Analogous to the notion of the *interval of oscillation* as the distance between successive zeros of a solution of an equation of the second order, is the notion of *forward* and *backward intervals of oscillation.*

Consider the totality of those solutions of (1) which vanish at $x = a$; these solutions will form a two parameter linear family. Suppose that they all vanish again in the interval (a, b). The *least* interval (a, β), where β is greater than a, which has the property that all these solutions vanishing at $x = a$ will vanish again in (a, β) is called the *forward interval of oscillation* at $x = a$. Likewise the *least* interval (β, a) where β is less than a, which has the property that all these solutions vanish again in (β, a) is called the *backward interval of oscillation* at $x = a$.

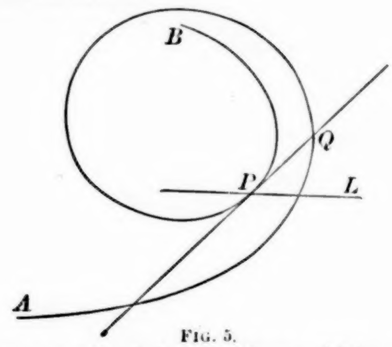

Fig. 5.

It may happen that not all of the solutions which, vanish for $x = a$ vanish again for $x > a$; in this case a forward interval of oscillation at $x = a$ does not exist. Likewise a backward interval of oscillation may not exist.

Let us now consider a regular interval (a, b) which, as before, is taken to be of the first kind. The integral curve AB will then be a spiraliform curve with the point B in its interior (see Fig. 5). For such an interval (a, b) a

forward interval of oscillation never exists, unless AB is an oval or touches itself. Thus in the figure let P be any point whose x is a; then the line PL corresponds to a solution which does not vanish for $x > a$, since PL does not meet PB at any second point.

The backward interval of oscillation may however exist. Thus in the figure, the tangent to the curve AB at P first meets the arc AP at Q; if the point Q corresponds to $x = \beta$, then (β, a) is the backward interval of oscillation at a. Any straight line through P meets QP again and this is not true of any shorter arc PQ'. Since $\phi(x,a)$ corresponds to the tangent at a, the quantity β is the greatest root of $\phi(x,a) = 0$ less than a. If the tangent at P does not meet AP again, the backward interval of oscillation does not exist.

It is apparent from this figure that any solution vanishes at most in two distinct points in a backward interval of oscillation (β, a) on such an interval (a, b). Furthermore, if a given solution y vanishes for any value of $x > a$, it will necessarily vanish in (β, a). This is obvious from the figure also.

Similar facts are true of forward intervals of oscillation if (a, b) is of the second kind.

Comparison Theorem. *If (a, b) be a regular interval of the first (second) kind for the equation* (6), *and if $R \leqq 0$ ($R \geqq 0$), the backward (forward) interval of oscillation (β_1, a) $[(a, \beta_1)]$ at $x = a$ for* (12) *is smaller than the like interval (β, a) $[(a, \beta)]$ for* (6).

Proof. To prove this theorem we shall again employ the auxiliary formula (16). As in a preceding proof we shall take

$$\eta = \phi\,(x, a)$$

and thus obtain again (17). Suppose now if possible that the theorem is not true, so that we have $\beta \geqq \beta_1$. As we have already noted, the function analogous to $\phi(x, a)$ for the modified equation (12) is precisely y of (17), which must then vanish for $x = \beta_1$ but not for $\beta_1 < x < a$, by the analytic condition obtained above, since (a, b) is regular for (12) also. The two functions y and $\phi(x, a)$ will therefore be positive throughout (β, a) since neither vanishes within (β, a) and since both are positive in the vicinity of a by (11). Put now $x = \beta$ in equation (17). It reduces to

$$y(\beta) = \int_\beta^a \phi(\beta, \xi)\, R(\xi)\, y(\xi) d\xi,$$

where $\phi(\beta, \xi)$ is positive. But $R(\xi)$ is negative or zero. Therefore in this

relation are equated quantities, one positive or zero, the other actually negative. This is impossible and the theorem must hold.

We note the following result contained in the above theorem: *The backward (forward) interval of oscillation for the equation*

$$y''' + qy' + \left(\frac{q'}{2} + R\right) y = 0,$$

where $R \leqq 0$ ($R \geqq 0$) *is smaller than that for the self-adjoint equation obtained by taking* $R = 0$.

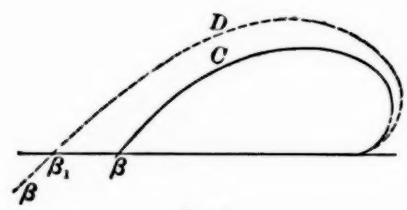

FIG. 6.

Any equation (1) may be reduced to the normal form above by a known transformation. In the present section we have developed the dependence of the solutions on R, by showing that if R is negative a decrease in R decreases the backward interval of oscillation. In the figure (Fig. 6) we have represented by a full line the integral curve C of this equation and by a broken line the integral curve D of the modified equation. If (β, a) represents the backward interval of oscillation at a for the curve C and (β_1, a) the similar interval for D, we have $\beta_1 > \beta$. For the sake of the comparison the curves have been taken with the same initial point and direction. It would now be desirable to give a comparison theorem which stated, in simple terms, the dependence of the interval of oscillation on the function q. If we increase q, however, it may be shown that the interval of oscillation may be either increased or diminished in length. In the self-adjoint case the dependence is more simple:

COMPARISON THEOREM. *Self-adjoint case. Given two equations reducible to self-adjoint form, in particular to*

$$(20) \qquad \begin{aligned} &y''' + q_1 y' + \frac{q_1'}{2} y = 0, \\ &y''' + q_2 y' + \frac{q_2'}{2} y = 0, \end{aligned}$$

such that $q_1 > q_2$ in (a, b); then the backward (forward) interval of oscillation of the first of these equations is less than that of the second.

Proof. In order to prove this theorem we shall prove first that the backward interval (β, a) of oscillation of any self-adjoint equation

$$y''' + qy' + \frac{q'(x)}{2}\,y = 0 \tag{21}$$

is precisely the interval of oscillation (β, a) for the linear differential equation of the second order

$$z'' + \frac{q}{4}\,z = 0. \tag{22}$$

Let y be the solution of the equation (21) such that

$$y(a) = y'(a) = 0, \qquad y''(a) = 1. \tag{23}$$

That is, y is chosen so that it is equal to $\phi(x, a)$ by (11). As has been shown, the backward interval (β, a) of oscillation at $x = a$ is given by the largest root of $y = 0$ which is less than a.

Now we have clearly, by triple integration by parts,

$$\int yy'''dx = yy'' - y'^2 + y''y - \int y'''y dx,$$

and by a single integration by parts,

$$\int y(qy')dx = qy^2 - \int (qy' + q'y)ydx.$$

Therefore we have

$$\int y(y''' + qy' + \frac{q'}{2}\,y)dx = 2yy'' - y'^2 + qy^2 - \int (y''' + qy' + \frac{q'}{2}\,y)y\,dx.$$

Remembering now that y is a solution of equation (21) which furthermore satisfies conditions (23) we obtain

$$2yy'' - y'^2 + qy^2 = 0.$$

Make the substitution

$$y = z^2, \quad y' = 2\,zz', \quad y'' = 2(zz'' + z'^2),$$

in this last equation, and, on the removal of a factor $4z^3$ it will reduce to the linear differential equation (22). It is clear that, as thus determined, z will be a solution which vanishes at $x = a$. And β will be determined of course as the greatest value of x less than a for which $z = 0$. *Hence the backward (forward) interval of oscillation at a of the given self-adjoint equation* (21) *is given as an interval between two successive zeros, of a solution of the associated linear differential equation* (22) *of the second order.*

The theorem we wish to prove is now an obvious consequence of the Comparison Theorem stated in the introduction.

The above reduction is not new.

7. Oscillatory and Non-Oscillatory Solutions. The preceding paragraphs afford practical methods for obtaining an idea as to the distribution of the zeros of a solution of (1) provided the solution is known to vanish. The following theorem often enables one to determine whether a given solution is oscillatory or non-oscillatory in the interval (a, b) i. e. whether the solution vanishes or not.

THEOREM. *Let* (a, b) *be a regular interval of the first (second) kind for the equation* (6), *and suppose that* $R \leqq 0$ $(R \geqq 0)$. *Let* η *and* y *be solutions of* (6) *and* (12) *respectively, so related that*

$$y(a) = \eta\,(a), \qquad y'(a) = \eta'\,(a), \qquad y''(a) = \eta''(a).$$

The inequality

$$|y| < |\eta| \tag{24}$$

obtains for values of x *less (greater) than* a *and within the backward (forward) interval of oscillation at* $x = a$ *until a value of* x *is reached for which* $y = 0$; *the inequality* $|y| > |\eta|$ *obtains for all values of* x *greater (less) than* a *until a value of* x *is reached for which* $\eta = 0$.

Proof. If the inequality (24) fails to hold for $x < a$, it will certainly fail to hold before an x is reached for which one has $\eta = 0$. In fact, otherwise, for this x we have $y = 0$, by (24), and the theorem does not say anything about y for lesser values of x. Therefore if the first part of the theorem does not hold, the inequality

$$|y| \geqq |\eta| \tag{25}$$

is true for some $x < a$, say x_0, in the backward interval of oscillation (β, a), such that neither y nor η vanish in (x_0, a).

But y and η have precisely the properties that they had in the auxiliary formula (16) so that

$$y(x_0) - \eta(x_0) = \int_{x_0}^{a} \phi(x_0, \xi)\ R(\xi)\ y(\xi)\ d\xi.$$

The left hand member in this equality is of the same sign as $y(x_0)$ or zero, since (25) holds at $x = x_0$. The right hand member is of opposite sign to $y(x_0)$, since $R(\xi)$ is negative and $\phi(x_0, \xi)$ is positive, x_0 and ξ being contained within the interval of oscillation (β, a) (see Fig. 5). This equality cannot hold and the first part of the theorem is true.

The second part of the theorem is proved in an analogous way. An equation

$$y(x_0) - \eta(x_0) = -\int_{a}^{x_0} \phi(x_0, \xi)\ R(\xi)\ y(\xi)\ d\ \xi$$

will be obtained if the theorem is not true, in which $x_0 > a$ is an x for which $|y| \leqq |\eta|$ while neither y nor η change sign in (a, x_0). In this case too $R(\xi)$ is negative and $\phi(x_0, \xi)$ is positive. As before, the two members of this equality would have opposite signs.

The relation of the two solutions is given in the figure (Fig. 7).

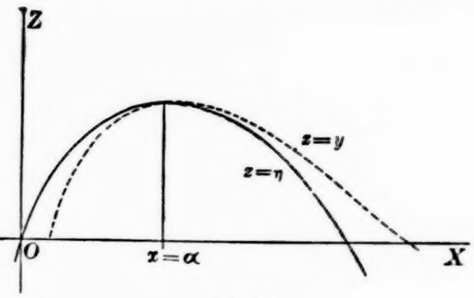

FIG. 7.

The following application is at once obvious: *If η and y are solutions of the equations* $y''' + qy' + \frac{q'}{2}\, y = 0$ *and of* $y''' + qy' + \left(\frac{q'}{2} + R\right) y = 0$ *respectively, where* $R \leqq 0$ $(R \geqq 0)$, *and are so related that*

$$y(a) = \eta\ (a),\quad y'(a) = \eta'(a),\quad y''(a) = \eta''(a),$$

then $|y| < |\eta|$ *for* $x < a (x > a)$ *and* $|y| > |\eta|$ *for* $x > a (x < a)$ *until a value of x is reached for which $y = 0$ and $\eta = 0$ respectively.*

In fact the interval (a, b) under consideration is regular and of either kind for the self-adjoint equation.

8. An Example. It is possible in many cases to form a good idea of the character of the solutions by means of the preceding results. As an illustration we consider the equation

$$y''' + y' + xy = 0,$$

to which the adjoint equation is

$$z''' + z' - xz = 0.$$

We choose y_1, y_2, y_3 as the principal solutions at $x = 0$:

$$\begin{array}{lll} y_1(0) = 0, & y_1'(0) = 0, & y_1''(0) = 1, \\ y_2(0) = 0, & y_2'(0) = 1, & y_2''(0) = 0, \\ y_3(0) = 1, & y_3'(0) = 0, & y_3''(0) = 0. \end{array}$$

The adjoint solutions (see equations (3) then satisfy the conditions

$$\begin{array}{lll} z_1(0) = 1, & z_1'(0) = 0, & z_1''(0) = -1, \\ z_2(0) = 0, & z_2'(0) = -1, & z_2''(0) = 0, \\ z_3(0) = 0, & z_3'(0) = 0, & z_3''(0) = 1. \end{array}$$

The interval $x < 0$ is regular and of the first kind (test for regular intervals, special case) and the interval $x > 0$ is regular and of the second kind. Each of the corresponding parts of the integral curve is therefore spiraliform with the point A corresponding to $x = 0$ in its interior.

Furthermore it is clear that y_1, y_3, z_1, z_3, are even functions of x and that y_2, z_2 are odd functions, since the equation is unchanged if $-x$ be substituted for x. The integral curve must be symmetrical with respect to $w_2 = 0$. Hence the integral curve has the form indicated in the schematic figure (Fig. 8).

The zeros of y_1 and y_2 and indeed of any two members of the linear family $c_1 y_1 + c_2 y_2$, will alternate for $x > 0$ and for $x < 0$ (Separation Theorem for Regular Intervals).

In order to consider the question of the distribution of the zeros of y_1,

y_2 for $x > 0$ we note that the forward interval of oscillation will decrease with increase of x and always be less than that of

$$y''' + y' = 0$$

which is precisely 2π (Comparison Theorem). Hence as x increases the successive zeros of y_1 and of y_2 come at intervals none exceeding 2π and tending to zero.

The function y_3 also vanishes for $x = x_1 > 0$, as a direct computation shows. The successive zeros of $y_3(x)$ will fall of course at intervals less than 2π which tend to zero as x increases.

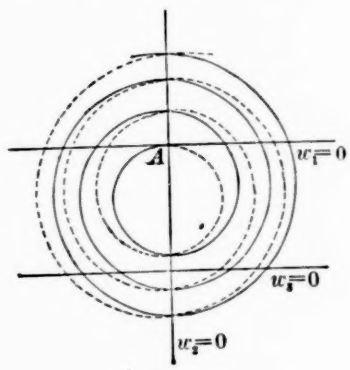

FIG. 8.

9. The Theorems of Liouville. The only known theorems of the kind we have been considering concerning the solution of the third (and higher) order equations appear to be those due to Liouville.*

The type of the equation considered is

$$\frac{d}{dx} K \frac{d}{dx} L \frac{d}{dx} y + \lambda y = 0,$$

where K, L are positive in (a, b) and λ is a parameter. Now when $\lambda = 0$ one finds by direct integration that

$$\phi(x, a) = \int_a^x \frac{c}{L} \int_a^x \frac{1}{K} dx.$$

* *Liouville's Journal*, vol. 3, pp. 561–614 (1838).

This function does not vanish for $x \neq a$. Hence for $\lambda = 0$ the interval (a, b) is of the first and second kind, the corresponding arc AB has the shape of a part of a convex oval. Suppose now that the equation is reduced to the form (6) and that λ is made to increase. By the test for regular intervals and the Comparison Theorem, the interval (a, b) remains regular of the second kind and the forward intervals of oscillation decrease. In addition, we know by the theorem of §7 that if we fix y, y', y'' at $x = a$, the first greater value of x for which $y = 0$ will decrease with increase of λ.

The results of Liouville are still more definite, such as the following: If we consider a particular solution $y(x)$ for which at $x = a$

$$y = a, \qquad \frac{d}{dx}y = b, \qquad \frac{d}{dx}L\frac{d}{dx}y = c, \qquad a > 0, \qquad b > 0, \qquad c > 0,$$

all of the greater values of x for which y vanishes must decrease with increase of λ.

This result is equivalent to his oscillation theorem which states that there exists one and only one value of λ, say λ_0, for which y vanishes at b and not within (a, b), and one and but one value λ_1, such that y vanishes at b and once within (a, b) and so on.

Liouville's results belong to another category than those which we have obtained. We have given only theorems concerned with any arbitrary increase of the coefficients. A special kind of variation was considered by Liouville which is however of great importance.

Princeton University,
June, 1910.

APPROXIMATE REPRESENTATION

BY W. E. BYERLY

1. Introduction. THAT the Least Square criterion employed in this paper could be made to give the received coefficients in a Fourier or in a Zonal Harmonic development appears to have been first noticed by Plarr who communicated his discovery through Bertrand to the *Comptes Rendus* in 1857.*

Later Toepler† and Gram‡ who were apparently unacquainted with Plarr's paper used practically the same method as Plarr; and these methods have been employed to some extent by recent writers on Integral Equations.§

So far as I have been able to discover these writers have usually confined their attention to the one-dimensional case, although some of them have mentioned that the methods they employed could be extended to the case of two-dimensional and three-dimensional problems.

That the methods here discussed furnish a very convenient and powerful tool in dealing with problems in Mathematical Physics which require the use of the so-called "Harmonic Analysis" is not, however, generally known, and I have found them so useful in my own work that it seemed to me worth while to communicate them to my fellow workers.

2. The Mean Square Criterion. Suppose that $f(x)$ and $F(x)$ are two given functions and that we wish to regard the second as an approximate representation of the first over a specified range of values for x, i.e., $x_0 < x < x_1$; is there a convenient criterion by which we can judge the excellence of the approximation?

When such a representation is used it is ordinarily desirable that the absolute value of the *error* $E \equiv f(x) - F(x)$, which of course is a function of x, should be small for all values of x considered. Of course in that case $E^2 \equiv [f(x) - F(x)]^2$ will be small and $[E^2]$, the *mean value* of E^2 over the range in question, will be small.

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* *Comptes Rendus*, vol. 44, p. 985.

† *Wiener Anzeige*, vol. 13, (1876), p. 205.

‡ *Crelle*, vol. 94, (1883), p. 41.

§ M. Bôcher, *An Introduction to the Study of Integral Equations*, pp. 52–60.

This $[E^2]$, the Mean Square of the Error, that is, the mean value of the square of the *error* $f(x) - F(x)$, over the range in question, we shall use as the measure of the excellence of our approximation. The smaller $[E^2]$, the better we shall call our representation.

Of course, since the mean of a set of positive quantities cannot be zero unless every one of the quantities is separately zero, our representation is *exact* if $[E^2] = 0$, and in that case $f(x)$ and $F(x)$ are equal for all values of x between x_0 and x_1.

It is easily seen that

$$\left[E^2\right] \equiv \frac{1}{x_1 - x_0} \int_{x_0}^{x_1} [f(x) - F(x)]^2\, dx. \qquad \text{[I]}$$

3. Determination of the "Best" Coefficients in an Approximate Representation of Specified Form. Let $f(x)$ and $\phi_1(x)$ be given functions* of x and let us try to determine a constant coefficient a_1 so that $a_1\, \phi_1(x)$ shall *represent* $f(x)$ over the range from x_0 to x_1 as well as possible.

The problem is a very simple one in *maxima and minima*.

$$\left[E^2\right] \equiv \frac{1}{x_1 - x_0} \int_{x_0}^{x_1} [f(x) - a_1\phi_1(x)]^2 dx,$$

regarded as a function of a_1, is to be made as small as possible. Write

$$\frac{d[E^2]}{da_1} \equiv -\frac{2}{x_1 - x_0} \int_{x_0}^{x_1} [f(x) - a_1\phi_1(x)]\, \phi_1(x)\, dx = 0\,;$$

we get
$$a_1 = \frac{C_1}{A_1}, \qquad \text{[II]}$$

where
$$A_1 = \int_{x_0}^{x_1} [\phi_1(x)]^2 dx$$

and
$$C_1 = \int_{x_0}^{x_1} f(x)\, \phi_1(x) dx\,;$$

then
$$\left[E^2\right] = \frac{1}{x_1 - x_0} \left[\int_{x_0}^{x_1} [f(x)]^2 dx - 2a_1 C_1 + a_1^2 A_1\right].$$

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* Throughout this paper we shall suppose that the functions employed are of a "respectability" above suspicion; finite, continuous, single-valued, integrable, and differentiable over the range or throughout the region considered. It is true that in many cases some of these limitations may be dispensed with, "but that is another story."

But from [II] $$2a_1C_1 = 2a_1^2A_1.$$

Hence

$$\left[E^2\right] = \frac{1}{x_1 - x_0}\left[\int_{x_0}^{x_1}[f(x)]^2dx - a_1^2A_1\right]. \qquad \text{[III]}$$

4. Having found a_1 let us now determine a_2 so that the addition of the term $a_2\,\phi_2(x)$, $\phi_2(x)$ being another given function, shall *improve* our representation as much as possible. To do this we have only to determine a_2 so that $a_2\,\phi_2(x)$ shall represent $f(x) - a_1\,\phi_1(x)$ as well as possible, and we get, by §3,

$$a_2 = \frac{\int_{x_0}^{x_1} f(x)\,\phi_2(x)dx - a_1\int_{x_0}^{x_1}\phi_1(x)\,\phi_2(x)dx}{\int_{x_0}^{x_1}[\phi_2(x)]^2dx},$$

or, if

$$A_1 = \int_{x_0}^{x_1}[\phi_1(x)]^2\,dx, \qquad A_2 = \int_{x_0}^{x_1}[\phi_2(x)]^2\,dx, \qquad C_1 = \int_{x_0}^{x_1}f(x)\,\phi(x)\,dx,$$

$$C_2 = \int_{x_0}^{x_1}f(x)\,\phi_2(x)dx, \text{ and } B_{1,2} = \int_{x_0}^{x_1}\phi_1(x)\phi_2(x)\,dx,$$

then

$$a_2 = \frac{C_2 - a_1B_{1,2}}{A_2}\,; \qquad \text{[IV]}$$

and

$$\left[E^2\right] = \frac{1}{x_1 - x_0}\left[\int_{x_0}^{x_1}[f(x)]^2dx - a_1^2A_1 - a_2^2A_2\right]. \qquad \text{[V]}$$

5. Let us go back and determine the coefficients a_1 and a_2 *ab initio* so that $a_1\,\phi_1(x) + a_2\,\phi_2(x)$ shall represent $f(x)$ as well as possible over the range x_0 to x_1.

Here we have a problem in *maxima* and *minima* of a function of two independent variables.

$$\left[E^2\right] \equiv \frac{1}{x_1 - x_0}\int_{x_0}^{x_1}[f(x) - a_1\phi_1(x) - a_2\phi_2(x)]^2\,dx\,;$$

write

$$\frac{\partial[E^2]}{\partial a_1} \equiv \frac{-2}{x_1 - x_0}\int_{x_0}^{x_1}[f(x) - a_1\phi_1(x) - a_2\phi_2(x)]\,\phi_1(x)dx = 0,$$

$$\frac{\partial[E^2]}{\partial a_2} \equiv \frac{-2}{x_1 - x_0}\int_{x_0}^{x_1}[f(x) - a_1\phi_1(x) - a_2\phi_2(x)]\,\phi_2(x)\,dx = 0.$$

To get a_1 and a_2 we have to solve the linear equations

$$A_1a_1 + B_{1,2}a_2 = C_1, \qquad [\mathrm{VI}]$$
$$B_{1,2}a_1 + A_2a_2 = C_2,$$

where the notation is the same as in §4.

$$\left[E^2\right] = \frac{1}{x_1 - x_0}\left[\int_{x_0}^{x_1}[f(x)]^2dx + a_1^2A_1 + a_2^2A_2 - 2a_1C_1 - 2a_2C_2 + 2a_1a_2B_{1,2}\right].$$

From [VI]
$$2a_1C_1 = 2a_1^2A_1 + 2a_1a_2B_{1,2},$$
$$2a_2C_2 = 2a_2^2A_2 + 2a_1a_2B_{1,2}.$$

Hence

$$\left[E^2\right] = \frac{1}{x_1 - x_0}\left[\int_{x_0}^{x_1}[f(x)]^2\,dx - a_1^2A_1 - a_2^2A_2 - 2a_1a_2B_{1,2}\right]. \qquad [\mathrm{VII}]$$

6. Example 1. (a) To determine a_1 so that a_1x shall be the best approximate representation of $\sin x$ from $x = 0$ to $x = \pi$.

Here

$$C_1 = \int_0^\pi x \sin x dx = \pi, \qquad A_1 = \int_0^\pi x^2dx = \frac{\pi^3}{3};$$

$$a_1 = \frac{C^1}{A_1} = \frac{3}{\pi^2} = 0.304;$$

and $\sin x = 0.304x$ approximately if $0 < x < \pi$.

As our test of the excellence of our approximation we have, since

$$\int_0^\pi \sin^2 xdx = \frac{\pi}{2},$$

$$\left[E^2\right] = \frac{1}{\pi}\left[\frac{\pi}{2} - \frac{9}{\pi^4}\frac{\pi^3}{3}\right] = \frac{1}{2} - \frac{3}{\pi^2} = 0.196.$$

It is interesting to note that should we take for $\sin x$ the first term of the familiar power series

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

we should have over the same range

$$\left[E^2\right] = \frac{\pi^2}{3} - \frac{3}{2} = 1.789.$$

(*b*) To determine a_2 so that the addition of the term a_2x^3 shall improve our representation as much as possible.

Here A_1 and C_1 have the same values as in (*a*).

$$A_2 = \frac{\pi^7}{7}, \qquad C_2 = \pi^3 - 6\pi, \qquad \text{and } B_{1,2} = \frac{\pi^5}{5};$$

$$a_2 = \frac{14}{\pi^4}\left[\frac{1}{5} - \frac{3}{\pi^2}\right] = -0.015;$$

and for

$$\sin x = 0.304x - 0.015x^3$$

$$\left[E^2\right] = \frac{1}{\pi}\left[\frac{\pi}{2} - \frac{\pi^3}{3}a_1^2 - \frac{\pi^7}{7}a_2^2\right] = 0.0307.$$

(*c*) To determine a_1 and a_2 so that $a_1x + a_2x^3$ shall represent $\sin x$ as well as possible over the range $0 < x < \pi$. Here A_1, A_2, C_1, C_2, and $B_{1,2}$ have the same values as in (*b*). We have

$$\frac{\pi^3}{3}a_1 + \frac{\pi^5}{5}a_2 = \pi, \qquad \frac{\pi^5}{5}a_1 + \frac{\pi^7}{7}a_2 = \pi^3 - 6\pi.$$

Hence

$$a_1 = \frac{15}{2\pi^2}\left(\frac{21}{\pi^2} - 1\right) = 0.8576,$$

$$a_2 = \frac{35}{2\pi^4}\left(1 - \frac{15}{\pi^2}\right) = -0.09348,$$

and

$$\sin x = 0.8576x - 0.09348x^3$$

approximately if $0 < x < \pi$. This last approximation is excellent since

$$\left[E^2\right] = \frac{1}{\pi}\left[\frac{\pi}{2} - \frac{\pi^3}{3}a_1^2 - \frac{\pi^7}{7}a_2^2 - 2\,\frac{\pi^5}{5}a_1a_2\right] = 0.005.$$

Should we use as our approximation

$$\sin x = x - \frac{x^3}{3!}$$

we should have

$$[E^2] = 0.4008$$

over the same range.

7. Example 2. (*a*) To determine a_1 so that $a_1 \sin x$ shall represent x over the range from 0 to π as well as possible.

Here

$$A_1 = \int_0^\pi \sin^2 x\, dx = \frac{\pi}{2}, \qquad C_1 = \int_0^\pi x \sin x\, dx = \pi,$$

$$a_1 = \frac{C_1}{A_1} = 2,$$

and

$$x = 2 \sin x$$

approximately $(0 < x < \pi)$.

As our test of the excellence of the approximation we have

$$\left[E^2\right] = \frac{1}{\pi}\left[\frac{\pi^3}{3} - 2\pi\right] = \frac{\pi^2}{3} - 2 = 1.29.$$

(*b*) To determine a_2 so that $a_2 \sin 3x$ shall improve the value just obtained as much as possible.

Here

$$A_2 = \int_0^\pi \sin^2 3x\, dx = \frac{\pi}{2},$$

$$C_2 = \int_0^\pi x \sin 3x\, dx = \frac{\pi}{3},$$

$$B_{1,2} = \int_0^\pi \sin x \sin 3x\, dx = 0,$$

so that

$$a_2 = \frac{2}{3};$$

here

$$\left[E^2\right] = \frac{1}{\pi}\left[\frac{\pi^3}{3} - 2\pi - \frac{2}{9}\pi\right] = \frac{\pi^2}{3} - 2 - \frac{2}{9} = 1.07.$$

(*c*) To determine a_1 and a_2 so that $a_1 \sin x + a_2 \sin 3x$ shall represent x as well as possible over the range $0 < x < \pi$. We have

$$\frac{\pi}{2} a_1 = \pi, \qquad \frac{\pi}{2} a_2 = \frac{\pi}{3},$$

so that

$$a_1 = 2, \qquad \text{and} \qquad a_2 = \frac{2}{3};$$

$$\left[E^2\right] = \frac{1}{\pi}\left[\frac{\pi^3}{3} - 2\pi - \frac{2}{9}\pi\right] = 1.07.$$

8. Example 3. To determine a_1 and a_2 so that $a_1 \sin x + a_2 \sin^3 x$ shall represent $\sin 3x$ as well as possible over the range $0 < x < \frac{\pi}{2}$.

$$A_1 = \int_0^{\pi/2} \sin^2 x\, dx = \frac{\pi}{4},$$

$$A_2 = \int_0^{\pi/2} \sin^6 x\, dx = \frac{5\pi}{32},$$

$$B_{1,2} = \int_0^{\pi/2} \sin x \sin^3 x\, dx = \frac{3\pi}{16},$$

$$C_1 = \int_0^{\pi/2} \sin 3x \sin x\, dx = 0,$$

$$C_2 = \int_0^{\pi/2} \sin 3x \sin^3 x\, dx = -\frac{\pi}{16}.$$

$$\frac{\pi}{4} a_1 + \frac{3\pi}{16} a_2 = 0, \qquad \frac{3\pi}{16} a_1 + \frac{5\pi}{32} a_2 = -\frac{\pi}{16}.$$

Hence

$$a_1 = 3,$$

$$a_2 = -4,$$

$$[E^2] = \frac{2}{\pi}\left[\frac{\pi}{4} - \frac{9\pi}{4} - 16 \cdot \frac{5\pi}{32} + 24 \cdot \frac{3\pi}{16}\right] = 0,$$

and the representation is exact, and

$$\sin 3x = 3 \sin x - 4 \sin^3 x$$

for all values of x between 0 and $\frac{\pi}{2}$.

9. The method of §5 is easily generalized. Let it be required to determine $a_1, a_2, \cdots, a_n$ so that

$$\sum_{k=1}^{k=n} a_k \phi_k(x)$$

shall represent $f(x)$ over the range $x_0 < x < x_1$ as well as possible.

We find that the coefficients can be obtained by solving the n linear equations

$$\sum_{k=1}^{k=l-1} B_{k,l} a_k + A_l a_l + \sum_{k=l+1}^{k=n} B_{k,l} a_k = C_l, \quad (l = 1, 2, 3, \cdots n.) \qquad \text{[VIII]}$$

$$\left[E^2\right] = \frac{1}{x_1 - x_0}\left\{\int_{x_0}^{x_1}[f(x)]^2\, dx - \sum_{k=1}^{k=n} A_k a_k^2 - 2\sum_{k=1}^{k=n-1}\sum_{l=k+1}^{l=n} B_{k,l}a_k a_l\right\}, \text{[IX]}$$

where

$$A_k = \int_{x_0}^{x_1}[\phi_k(x)]^2\, dx,$$

$$B_{k,l} = \int_{x_0}^{x_1}\phi_k(x)\ \phi_l(x)\ dx,$$

and

$$C_k = \int_{x_0}^{x_1} f(x)\ \phi_k(x)\ dx.$$

If we desire to correct our representation as much as possible by the addition of another term $a_{n+1}\ \phi_{n+1}\ (x)$,

$$a_{n+1} = \frac{C_{n+1} - \sum_{k=1}^{k=n} a_k\ B_{k,n+1}}{A_{n+1}}, \qquad \text{[X]}$$

and

$$\left[E_1^2\right] = \frac{1}{x_1 - x_0}\left\{\int_{x_0}^{x_1}[f(x)]^2\, dx - \sum_{k=1}^{k=n+1} A_k a_k^2 - 2\sum_{k=1}^{k=n-1}\sum_{l=k+1}^{l=n} B_{k,l}\, a_k\, a_l\right\},$$

or

$$[E_1^2] = [E^2] - A_{n+1}a_{n+1}^2. \qquad \text{[XI]}$$

10. Orthogonal Functions. A pair of functions the integral of whose product taken over the range $x_0 < x < x_1$ is zero we shall call orthogonal over the range.

That is if

$$\int_{x_0}^{x_1}\phi_m(x)\ \phi_n(x)\ dx = 0,$$

$\phi_m(x)$ and $\phi_n(x)$ are orthogonal over the range $x_0 < x < x_1$. A set of functions such that every pair is orthogonal over our range we shall call a *set of orthogonal functions* over the range.

If in §9 the functions $\phi_1(x)$, $\phi_2(x)$, $\cdots$ $\phi_{n+1}(x)$ are *orthogonal* our results simplify surprisingly.

Since

$$B_{k,l} = \int_{x_0}^{x_1}\phi_k(x)\ \phi_l(x)\ dx = 0$$

[VIII] gives at once

$$A_k = \frac{C_k}{A_k}; \qquad \text{[XII]}$$

[IX] gives

$$[E^2] = \frac{1}{x_1 - x_0}\left[\int_{x_0}^{x_1}[f(x)]^2\,dx - \sum_{k=1}^{k=n} A_k a_k^2\right]; \qquad \text{[XIII]}$$

and [X] gives

$$a_{n+1} = \frac{C_{n+1}}{A_{n+1}}, \qquad \text{[XIV]}$$

while [XI],

$$[E_1^2] = [E^2] - A_{n+1}\,a_{n+1}^2,$$

remains unchanged.

We see then that if we are trying to represent $f(x)$ as well as possible by the aid of a set of functions which are *orthogonal* over the range considered the coefficient a_k of a term $a_k\,\phi_k(x)$ which would be the best one if we used that term alone is the best one if that term is used in combination with any others of the set, and is the best one if that term is to be added as a correction to terms of the set whose best coefficients had been previously determined. Moreover since $[E^2]$ is necessarily positive no matter how many terms are used in the represention it is clear from formula (XIII) that the addition of any term with its appropriate coefficient* if the term can form an orthogonal pair with every term already present diminishes the Mean Square of the Error and improves the representation.

11. Corollaries. A few easy corollaries are:

(*a*) If $f(x)$ is capable of being exactly expressed in terms of the assumed functions for all values of x between x_0 and x_1 formula [VIII] §9 applied to the range $x_0 < x < x_1$ or to any portion of that range will give the correct coefficients. If it has not been foreseen that exact expression is possible formula [IX] will establish the fact by showing that $[E^2] = 0$. (Compare §8, Ex. 3).

(*b*) If it is known that for all values of x in a given range $f(x)$ can be expressed as a series of terms of the assumed form, and all the functions

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* It must be kept in mind, however, that the appropriate coefficient C_k/A_k may happen to be zero, in which case of course the improvement is null.

$\phi_k(x)$ employed are *orthogonal* over the range, formula [XII] §10 will give the correct coefficients of the series.

(*c*) No development of $f(x)$ in a series whose terms are *orthogonal* over the range over which the development is to hold good can be correct unless the set of forms $\phi_k(x)$ employed is a *complete orthogonal* set; that is unless all forms are present which might be inserted without preventing the whole set employed from being *orthogonal.**

(*d*) Since in computing the value of a function by the aid of a series into which it has been developed one can use only a limited number of terms, the check given by the value of $[E^2]$ is as valuable in practical applications in Physics when a valid series is employed for $f(x)$ as when an approximate representation is used.

(*e*) If in all that has gone before we replace *function* by *point-function*, *range* by *region* (one-, two-, or three-dimensional), and *integral over the range* by *integral throughout the region*, none of our formulas nor of our results will be changed in any essential particular.

12. Trigonometric Series. Since sines and cosines of whole multiples of x form a set of functions *orthogonal* over the range $-\pi < x < \pi$ the coefficients of the terms of Fourier type that best represent $f(x)$ over that range are given by formula [XII] §10.

But these are the familiar Fourier Coefficients.

Let $f(x) = \frac{1}{2}b_0 + b_1 \cos x + b_2 \cos 2x + \cdots + a_1 \sin x + a_2 \sin 2x + \cdots$

Then
$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos mx\, dx,$$

and
$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx\, dx.$$

$$[E^2] = \frac{1}{2\pi}\left\{\int_{-\pi}^{\pi}[f(x)]^2\,dx - \pi\sum(a_m^2 + b_m^2)\right\}.$$

Of course if $f(x)$ is developable in a Fourier's Series our coefficients a_m and b_m are the correct coefficients (v. corollary (*b*) §11).

13. Let us obtain and test a few Trigonometric expressions for x over the range $0 < x < \pi$.

* Apparent exceptions to this corollary are due to the fact that the appropriate coefficient C_k/A_k for a term of proposed form may happen to be zero.

Here, since when m and n are integers $\int_0^\pi \sin mx \sin nx\, dx = 0$, and $\int_0^\pi \cos mx \cos nx\, dx = 0$, while $\int_0^\pi \sin mx \cos nx\, dx$ is not generally zero, we can deal conveniently with a sine expression

$$x = \sum a_m \sin mx,$$

or a cosine expression

$$x = \sum b_m \cos mx.$$

These are

$$x = 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right], \tag{1}$$

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]. \tag{2}$$

For (1), $\quad [E^2] = \frac{1}{\pi}\left[\frac{\pi^3}{3} - 2\pi\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)\right] = 0,$

since $\quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

(v. Chrystal's Algebra vol. II, page 243), and as is well known (1) is a valid development of x over the range $0 < x < \pi$.

For (2)

$$[E^2] = \frac{1}{\pi}\left[\frac{\pi^3}{3} - \frac{\pi^3}{4} - \frac{16}{\pi^2}\frac{\pi^2}{2}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right)\right] = 0,$$

since $\quad \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$

(v. Chrystal's Algebra, vol. II, page 243), whence

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96};$$

and as is well known (2) is a valid development of x.

$$\int_0^\pi \sin mx \cos nx\, dx = 0$$

if $m+n$ is even, that is if m and n are both odd or both even. Sines and cosines of odd multiples of x form, then, a set of functions *orthogonal* over the range $0<x<\pi$, as do sines and cosines of even multiples of x. We can get

$$x=2\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\cdots\right)-\frac{4}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right), \quad (3)$$

and
$$x=\frac{\pi}{2}-2\left(\frac{\sin 2x}{2}+\frac{\sin 4x}{4}+\frac{\sin 6x}{6}+\cdots\right). \quad (4).$$

For (3),

$$[E^2]=\frac{1}{\pi}\left[\frac{\pi^3}{3}-2\pi\left(\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right)-\frac{8}{\pi}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right)\right]=0,$$

since
$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}.$$

For (4)

$$[E^2]=\frac{1}{\pi}\left[\frac{\pi^3}{3}-\frac{\pi^3}{4}-2\pi\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots\right)\right]=0,$$

and (3) and (4) are valid developments of x from $x=0$ to $x=\pi$, which are not usually given in the texts.

14. Spherical Harmonics. Our method gives the familiar coefficients in the case of the other *Harmonic* developments.

(*a*) *Zonal Harmonics*. For instance since

$$\int_{-1}^{1}P_m(x)\;P_n(x)\;dx=0$$

if m and n are integers, the Zonal Harmonics $P_m(x)$ form a set of functions *orthogonal* over the range $-1<x<1$. If we wish to express $f(x)$ as

$$\sum a_m\;P_m\;(x),$$

$$a_m=\frac{\int_{-1}^{1}f(x)\;P_m(x)dx}{\int_{-1}^{1}[P_m(x)]^2dx}=\frac{2m+1}{2}\int_{-1}^{1}f(x)\;P_m(x)\;dx,$$

and

$$[E^2]=\frac{1}{2}\left[\int_{-1}^{1}[f(x)]^2dx-\sum\frac{2}{2m+1}\,a^2{}_m\right].$$

The Zonal Harmonics $P_m(\cos\theta)$ are *orthogonal* over the surface of the unit sphere. (Byerly's *Fourier's Series* Art. 92).

If we wish to express $F(\theta)$ in terms of Zonal Harmonics and to use the unit sphere as our region we have

$$C_k = \int_0^{2\pi}\int_0^{\pi} F(\theta)\ P_k(\cos\theta)\ \sin\theta\ d\theta\ d\phi = 2\pi\int_0^{\pi} F(\theta)\ P_k(\cos\theta)\ \sin\theta\ d\theta.$$

$$A_k = \int_0^{2\pi}\int_0^{\pi}[P_k(\cos\theta)]^2\sin\theta\ d\theta\ d\phi = 2\pi\left[\frac{2}{2k+1}\right],$$

and
$$a_k = \frac{C_k}{A_k},$$
the familiar value. Here

$$[E^2] = \frac{1}{4\pi}\left\{2\pi\int_0^{\pi}[F(\theta)]^2\sin\theta\,d\theta - 4\pi\sum\frac{a_k^2}{2k+1}\right\}.$$

(*b*) *Tesseral Harmonics*. The Tesseral Harmonics of both types, i.e., $\cos n\phi P_m^n(\mu)$ and $\sin n\phi P_m^n(\mu)$, are a set of functions *orthogonal* over the unit sphere, unless in a pair of the same type the same subscript and the same index occur. (Byerly's *Fourier's Series*, Art. 105, and Art. 105, Ex. 2).

Hence if
$$a_{n,m} = \frac{C_{n,m}}{A_{n,m}},$$
where
$$C_{n,m} = \int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)\cos n\phi\ P_m^n(\mu)\ \sin\theta\ d\theta\ d\phi\ ,$$

$$A_{n,m} = \int_0^{2\pi}\int_0^{\pi}\cos^2 n\phi[P_m^n(\mu)]^2\sin\theta\ d\theta\ d\phi = \frac{2\pi}{2m+1}\,\frac{(m+n)!}{(m-n)!},$$

and
$$b_{n,m} = \frac{C'_{n,m}}{A'_{n,m}},$$
where
$$C'_{n,m} = \int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)\sin n\phi\ P_m^n(\mu)\ \sin\theta\ d\theta\ d\phi,\quad A'_{n,m} = A_{n,m},$$

and
$$a_{0,m} = \frac{C_{0,m}}{A_{0,m}},$$

where

$$C_{0,m} = \int_0^{2\pi}\int_0^{\pi} f(\theta, \phi)\, P_m(\mu) \sin\theta\, d\theta\, d\phi, \qquad A_{0,m} = \frac{4\pi}{2n+1},$$

then

$$f(\theta, \phi) = \sum_{m=0}^{m=\infty} \left[a_{0,m} P_m(\mu) + \sum_{n=1}^{n=m} (a_{n\,m} \cos n\phi + b_{n,m} \sin n\phi)\, P_m^n(\mu) \right],$$

(v. Byerly's *Fourier's Series*, Art. 107).

Here

$$[E^2] = \frac{1}{4\pi} \left\{ \int_0^{2\pi}\int_0^{\pi} [f(\theta, \phi)]^2 \sin\theta\, d\theta\, d\phi \right.$$

$$\left. - \sum_{m=0}^{m=\infty} \left[\frac{4\pi}{2m+1} a^2{}_{,m} + \sum_{n=1}^{n=m} \frac{2\pi}{2m+1} \frac{(m+n)\,!}{(m-n)\,!} \left(a_{n,m}^2 + b_{n,m}^2 \right) \right] \right\}.$$

15. Miscellaneous Examples. The examples which follow will give some notion of the convenience and power of the method of this paper in dealing with problems that require the use of the *Harmonic Analysis*.

In each of the first three examples a development of unity of importance in problems in Mathematical Physics is obtained and verified either directly or, as in Example 3, incidentally in the solution of an actual Physical Problem.

The developments in Example 1 and Example 2 are familiar, but are usually obtained by other methods, that in Example 3 I believe is new.

In Example 4 an approximate representation of unity in Bessel's Functions which are not *orthogonal* is considered.

In all the examples I refer freely to my *Fourier's Series* and *Spherical Harmonics*. (Boston, Ginn & Co.)

16. Example 1. To express 1 as

$$\sum \sum a_{m,n} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}.*$$

Here our functions $\phi_{m,n}\,(x,y)$ are easily seen to be *orthogonal* over the rectangle bounded by the axes and the lines $x = a$, $y = b$.

* v. *Fourier's Series*, pp. 127, 128.

$$C_{m,n} = \int_0^a \int_0^b \sin \frac{n\pi x}{a} \sin \frac{n\pi y}{b}\, dy\, dx = 0,$$

unless m and n are both odd, in which case

$$C_{m,n} = \frac{4ab}{mn\pi^2}.$$

$$A_{m,n} = \int_0^a \int_0^b \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b}\, dy\, dx = \frac{ab}{4}.$$

Hence $1 = \dfrac{16}{\pi^2}\left[\sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty} \dfrac{1}{(2m+1)(2n+1)} \sin \dfrac{(2m+1)\pi x}{a} \sin \dfrac{(2n+1)\pi y}{b}\right].$

$$\left[E^2\right] = \frac{1}{ab}\left[ab - \frac{ab}{4}\,\frac{25b}{\pi^2} \sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty} \frac{1}{(2m+1)^2(2n+1)^2}\right];$$

but the double series in the last term is easily seen to be the square of the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots,$$

that is the square of $\pi^2/8$, and $[E^2] = 0$ and our expression is a correct development of 1 holding good for $0 < x < a$ and $0 < y < b$.

17. Example 2. The Bessel's Functions $J_0(\mu_k r)$, $J_0(\mu_l r)$ are orthogonal over the area of the circle whose centre is the origin and whose radius is a if and only if $\mu_k a$ and $\mu_l a$ are roots of the equation $J_0(x) = 0$, or of $J_1(x) = 0$, or of $xJ_1(x) - \lambda J_0(x) = 0$. (*Fourier's Series*, Art. 125.)

(*a*). Let us express 1 as $\Sigma a_k\, J_0(\mu_k r)$ if $\mu_k a$ is a root of $J_0(x) = 0$.

Here $$a_k = \frac{C_k}{A_k},$$

where

$$C_k = \int_0^{2\pi}\int_0^a J_0(\mu_k r)\, r dr d\phi = 2\pi \int_0^a J_0(\mu_k r) r dr = \frac{2\pi a}{\mu_k} J_1(\mu_k a).$$

(*Fourier's Series*, p. 229 (6).)

$$A_k = \int_0^{2\pi}\int_0^a \Big[J_0(\mu_k r)\Big]^2 r dr d\phi = 2\pi\int_0^a \Big[J_0(\mu_k r)\Big]^2 r dr = \pi a^2\Big[J_1(\mu_k a)\Big]^2,$$

and
$$a_k = \frac{2}{\mu_k a J_1(\mu_k a)};$$

and we get
$$1 = 2\sum \frac{J_0(\mu_k r)}{\mu_k a J_1(\mu_k a)}. \tag{1}$$

Here
$$\Big[E^2\Big] = \frac{1}{\pi a^2}\left[\pi a^2 - \sum \frac{4\pi a^2[J_1(\mu_k a)]^2}{(\mu_k a)^2[J_1(\mu_k a)]^2}\right]$$
$$= \frac{1}{\pi a^2}\left[\pi a^2 - 4\pi a^2 \sum \frac{1}{(\mu_k a)^2}\right].$$

The equation
$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots = 0$$

may be regarded as an equation in x^2, and the sum of the reciprocals of its roots is the negative of the coefficient of its second term. Hence

$$\sum \frac{1}{(\mu_k a)^2} = \frac{1}{4}$$

and $[E^2] = 0$; and (1) is a correct development of 1 over the circle whose radius is a and therefore over the range $0 < r < a$.

(*b*) If $\mu_k a$ is a root of the equation $xJ_0(x) - \lambda J_1(x) = 0$, we have (v. *Fourier's Series*, p. 229)

$$C_k = \frac{2\pi a}{\mu_k} J_1(\mu_k a),$$

$$A_k = \frac{\pi(\lambda^2 + \mu_k^2 a^2)[J_0(\mu_k a)]^2}{\mu_k^2},$$

and
$$a_k = \frac{2\mu_k a \qquad J_1(\mu_k a)}{(\lambda^2 + \mu_k^2 a^2)[J_0(\mu_k a)]^2} = \frac{2\lambda}{(\lambda^2 + \mu_k^2 a^2)J_0(\mu_k a)};$$

$$1 = 2\lambda \sum \frac{J_0(\mu_k r)}{(\lambda^2 + \mu_k^2 a^2)J_0(\mu_k a)}. \tag{2}$$

$$\left[E^2\right] = \frac{1}{\pi a^2}\left[\pi a^2 - 4\lambda^2 \sum \frac{1}{(\lambda^2 + \mu_k^2 a^2)^2 [J_0(\mu_k a)]^2} \frac{\pi\ (\lambda^2 + \mu_k^2 a^2)[J_0(\mu_k a)]^2}{\mu_k^2}\right]$$

$$= \frac{1}{\pi a^2}\left[\pi a^2 - 4\pi a^2 \sum \frac{\lambda^2}{(\lambda^2 + \mu_k^2 a^2)\mu_k^2 a^2}\right].$$

$$\frac{\lambda^2}{(\lambda^2 + \mu_k^2 a^2)\mu_k^2 a^2} = \frac{1}{(\mu_k a)^2} - \frac{1}{\lambda^2 + \mu_k^2 a^2}.$$

The equation

$$xJ_1(x) - \lambda J_0(x) = -\lambda + \left(\frac{1}{2} + \frac{\lambda}{4}\right)x^2 - \left(\frac{1}{2^4} + \frac{\lambda}{2^2 4^2}\right)x^4 + \cdots = 0$$

may be regarded as an equation in x^2.

The sum of the reciprocals of its roots is the negative of the coefficient of the second term divided by the first term or $(2 + \lambda)/(4\lambda)$; that is,

$$\sum \frac{1}{\mu_k^2 a^2} = \frac{2 + \lambda}{4\lambda}.$$

To get

$$\sum \frac{1}{\lambda^2 + \mu_k^2 a^2}$$

is not so easy.

We must first transform $F(x^2) = xJ_1(x) - \lambda J_0(x) = 0$ into an equation whose roots exceed those of $F(x^2) = 0$ by λ^2.

The constant term will be $F(-\lambda^2)$ and the coefficient of the term of first degree will be $F'(-\lambda^2)$. (v. Todhunters' *Theory of Equations*, §54.).

Here $$F'(x^2) = \frac{1}{2x}\left[xJ_0(x) + \lambda J_1(x)\right] = \frac{1}{2}\left[J_0(x) + \frac{\lambda J_1(x)}{x}\right].$$

$$\sum \frac{1}{\lambda^2 + \mu_k^2 a^2} = -\frac{F'(-\lambda^2)}{F(-\lambda^2)} = -\frac{J_0(\lambda i) + \dfrac{\lambda J_1(\lambda i)}{\lambda i}}{2[\lambda i J_1(\lambda i) - \lambda J_0(\lambda i)]}$$

$$= -\frac{1}{2\lambda}\,\frac{J_0(\lambda i) - iJ_1(\lambda i)}{iJ_1(\lambda i) - J_0(\lambda i)} = \frac{1}{2\lambda}.$$

$$\sum\left[\frac{1}{\mu_k^2 a^2} - \frac{1}{\lambda^2 + \mu_k^2 a^2}\right] = \frac{2 + \lambda}{4\lambda} - \frac{1}{2\lambda} = \frac{1}{4}.$$

Hence $[E^2] = 0$ and (2) is a correct development of 1 over the circle whose radius is a, and therefore over the range $0 < r < a$.

18. Example 3. A homogeneous cylinder of radius c and altitude b is initially at the constant temperature unity throughout. Its entire surface is suddenly cooled to the temperature zero and kept there. Required the temperature of any point in the cylinder at any time after cooling has begun.

We have to satisfy the heat conduction equation

$$D_t u = a^2[D_r^2 u + \frac{1}{r} D_r u + D_z^2 u] \tag{1}$$

subject to the conditions

$$u = 0 \text{ when } z = 0 \tag{2}$$
$$u = 0 \text{ when } z = b \tag{3}$$
$$u = 0 \text{ when } r = c \tag{4}$$
$$u = 1 \text{ when } t = 0 \tag{5}$$

It is not difficult to find and it is easy to verify the particular solution

$$u = e^{-(\nu^2 + \mu^2)a^2 t} J_0(\mu r) \sin \nu z,$$

where μ and ν are any constants. This value satisfies equation (1) and condition (2).

If $\nu = \dfrac{m\pi}{b}$ it satisfies (3). If $\mu = \mu_k$, when $\mu_k c$ is a root of $J_0(x) = 0$, it satisfies (4).

Let $$u = \sum\sum a_{k,m} e^{-\left(\frac{m^2\pi^2}{b^2} + \mu_k^2\right)a^2 t} J_0(\mu_k r) \sin \frac{m\pi z}{b}.$$

This will be the required solution if

$$1 = \sum\sum a_{k,m} J_0(\mu_k r) \sin \frac{m\pi z}{b}$$

throughout the cylinder.

The functions $J_0(\mu_k r) \sin \dfrac{m\pi z}{b}$ are easily seen to be orthogonal throughout the cylinder.

$$a_{k,m} = \frac{C_{k,m}}{A_{k,m}},$$

where

$$C_{k,m} = \int_0^b \int_0^{2\pi} \int_0^c J_0(\mu_k r) \sin \frac{m\pi z}{b} \, r \, dr \, d\phi \, dz$$

$$= \frac{2cb}{m\mu_k} (1 - \cos m\pi) \, J_1(\mu_k c) \, ; \quad (\textit{Four. Ser.}, \text{ page } 229.)$$

$$A_{k,m} = \frac{\pi c^2 b}{2} [J_1(\mu_k c)]^2 ; \qquad (\textit{Four. Ser.}, \text{ page } 229.)$$

$$a_{k,m} = \frac{4(1 - \cos m\pi)}{m\pi \, \mu_k c \, J_1(\mu_k c)} \, ;$$

and our expression for 1 is

$$1 = \frac{4}{\pi} \sum \sum \frac{1 - \cos m\pi z}{m\mu_k c \, J_1(\mu_k c)} \, J_0(\mu_k r) \sin \frac{m\pi z}{b}.$$

$$\left[E^2\right] = \frac{1}{\pi c^2 b} \left[\pi c^2 b - \sum \sum \frac{8bc^2(1 - \cos m\pi)^2}{m^2\pi \, \mu_k c^2}\right].$$

$$\sum_{m=1}^{m=\infty} \frac{(1 - \cos m\pi)^2}{m^2\pi} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{4}{\pi}\,\frac{\pi^2}{8} = \frac{\pi}{2}$$

$$\sum_{k=1}^{k=\infty} \frac{1}{(\mu_k c)^2} = \frac{1}{4} \qquad (\text{v. Ex. } 2 \; (a) \; \S 17).$$

Hence $[E^2] = 0$, and our expression for 1 is exact and

$$u = \frac{8}{\pi} \sum_{m=0}^{m=\infty} \sum_{k=1}^{k=\infty} \frac{1}{(2m+1) \, \mu_k c \, J_1(\mu_k c)} \, J_0(\mu_k r) \sin \frac{(2m+1)\pi z}{b} \, e^{-\left(\frac{m^2\pi^2}{b^2} + \mu_k^2\right)a^2 t}$$

rigorously.

19. Example 4. In the preceding examples the functions in terms of which we have had occasion to express unity have in every case been orthogonal over the range or throughout the region considered. As an example of a different sort suppose we wish to express 1 in terms of Bessel's Functions $J_0(\mu_k x)$ which are not orthogonal over the circle over which the expression is to be used.*

* For a case of this sort see a paper by B. O. Peirce, *Proc. Amer. Acad.*, vol. 43, No. 5, Sept. 1907, p. 175, pp. 180-184.

There is no serious difficulty in getting any specified number of terms by the formulas of §6 of this paper.

We have

$$C_l = \int_0^{\pi}\int^{a} J_0(\mu_l r) r\, dr\, d\phi = \frac{2\pi a}{\mu_l} J_1(\mu_l a)\ ;$$

$$A_l = \int_0^{2\pi}\int_0^{a} [J_0(\mu_l r)]^2\, r\, dr\, d\phi = \pi a^2 \left\{ [J_0(\mu_l a)]^2 + [J_1(\mu_l a)]^2 \right\}\ ;$$

$$B_{k,l} = \int_0^{2\pi}\int_0^{a} J_0(\mu_k r)\ J_0(\mu_l r)\ r\, dr\, d\phi$$

$$= \frac{1}{\mu_k^2 - \mu_l^2}\left[\mu_k a\ J_0(\mu_l a)\ J_1(\mu_k a) - \mu_l a\ J_0(\mu_k a)\ J_1(\mu_l a)\right].$$

(v. *Fourier's Series*, §§125-126, pp. 228-229.)

For instance, if a single term $a_1 J_0(\mu_1 r)$ will suffice,

$$a_1 = \frac{C_1}{A_1} = \frac{2J_1(\mu_1 a)}{\mu_1 a\left[\left(J_0(\mu_1 a)\right)^2 + \left(J_1(\mu_1 a)\right)^2\right]}.$$

$$\left[E^2\right] = 1 - \frac{4[J_1(\mu_1 a)]^2}{\mu_1^2 a^2\left[\left(J_0(\mu_1 a)\right)^2 + \left(J_1(\mu_1 a)\right)^2\right]}.$$

If we need three terms we have

$$\begin{aligned} A_1 a_1 + B_{1,2} a_2 + B_{1,3} a_3 &= C_1, \\ B_{1,2} a_1 + A_2 a_2 + B_{2,3} a_3 &= C_2, \\ B_{1,3} a_1 + B_{2,3} a_2 + A_3 a_3 &= C_3, \end{aligned}$$

and

$$\left[E^2\right] = \frac{1}{\pi a^2}\left[\pi a^2 - A_1 a_1^2 - A_2 a_2^2 - A_3 a_3^2 - 2\left(B_{1,2} a_1 a_2 + B_{2,3} a_2 a_3 + B_{1,3} a_1 a_3\right)\right].$$

20. Conclusion. It may be objected that the test of excellence used in this paper is a purely artificial one. It is, however, often a very useful one and for many purposes it is precisely what is wanted ; and it is in fact the test that a specified number of terms in any of the received Harmonic developments always fulfil.

The method given here like other rough and ready methods, for instance that of Lagrange (*Fourier's Series*, §§19-25, *et passim*), has nothing to do, except indirectly, with the question of development in series. Indeed if a development in series of the proposed form is possible the coefficients of a specified set of terms given by the method of Least Squares are not the coefficients of the corresponding terms of the series unless the functions happen to be orthogonal throughout the region considered. In other cases they give a closer approximation than would the actual terms of the series. (v. Ex. 1 (*c*) §6.)

The check on the accuracy of the representation in any concrete case given by the value of $[E^2]$ is useful even when terms of a true series development are employed and it is easily applied in any case since all the constants in $[E^2]$ must have been calculated already in getting the coefficients.

If the functions used in the representation of $f(x)$ are orthogonal throughout the region in question the labor of getting the coefficients is much lessened, since every one is obtained separately, and every additional one obtained adds to the accuracy of the result.

If the functions employed are not orthogonal throughout the region the coefficients obtained are for many purposes the best possible, and whether or not the representation is accurate enough for the matter in hand is at once shown by the value of $[E^2]$.

In the subjects which gave rise to the Harmonic Analysis, i.e., in the Motions of Elastic Strings and in the Conduction of Heat, most of the developments required are in terms of functions orthogonal over the range or throughout the region concerned, and the old theory worked well enough. In problems in Electricity and Magnetism, and in Hydromechanics similar developments where the functions are not orthogonal are often desired and to them the old theory gives no clue.

Although our method has nothing to do with development in series except indirectly, corollaries (*b*) and (*c*) §11 show that it may often be of great service in that connection, and they suggest some beautiful little problems for the workers in the science of Integral Equations.

Harvard University,
Cambridge, Mass.,
July, 1910.

NOTE ON CUBIC EQUATIONS AND CONGRUENCES

L. E. Dickson

In the Annals for January, 1910, Mr. Escott has given certain cubic functions, for which the corresponding algebraic equations have either three* real roots or no real root, while the corresponding congruences have either three integral roots or no integral root. He did not, however, determine in what cases the roots of the congruences are integral and in what cases not integral. This problem will be treated here by two methods. Although the methods apply equally well to the other congruences, we restrict attention to that on page 90, which takes the following simpler form when x is replaced by $z-a$:

$$z^3 - 2az^2 + (2a-3)z + 1 \equiv 0 \pmod{p}. \tag{1}$$

In no case is $z = 0$ or $z = 1$ a root. Hence we may set

$$2a \equiv f(z) = \frac{z^3 - 3z + 1}{z^2 - z} \pmod{p}. \tag{2}$$

If y and z are roots of (1), then $f(z) = f(y)$. After clearing of fractions and factoring the resulting integral function, we get

$$(y-z)(yz-y+1)(yz-z+1) = 0.$$

Hence the values of y are

$$z, \quad \frac{1}{1-z}, \quad \frac{z-1}{z}. \tag{3}$$

If any two of these are equal, then all three are equal and

$$z^2 - z + 1 = 0. \tag{4}$$

The roots of (4) are $-\omega$ and $-\omega^2$, where $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{-3}$ is a cube root of unity. If $p = 3$, the only case in which (1) has an integral root is $a \equiv 0$ and

* It should be pointed out that the three roots may coincide and give a triple root.

the root -1 is then a triple root. Next, let -3 be a quadratic non-residue of p. Then (4) has no integral root and the functions (3) are distinct. Hence the $p-2$ integers z, where $1 < z < p$, give in sets of three the same value to (2). Thus there are $\frac{1}{3}(p-2)$ values of $2a$ for which congruence (1) has three (distinct) integral roots and $\frac{1}{3}(2p+2)$ values for which it has no integral root. For example, if $p=5$, it has integral roots (namely, 2, 3, 4) only when $2a \equiv 4$. Finally, let -3 be a quadratic residue of p. When z is a root of (4), $z^3=-1$ and $2a=3z$, so that $2a=-3\omega$ or $-3\omega^2$. The $p-4$ integers z, not roots of (4) and not 0 or 1, give in sets of three the same value to (2). Thus there are $2+\frac{1}{3}(p-4)$, namely $\frac{1}{3}(p+2)$, congruences (1) having integral roots, and $\frac{1}{3}(2p-2)$ congruences (1) having no integral root. For $2a \equiv -3\omega$, (1) has the triple root $-\omega$. For example, if $p=7$, it has the triple root 3 if $2a \equiv 2$, the triple root -2 if $2a \equiv 1$, the roots $-1, 2, 4$ if $2a \equiv -2$, but no integral roots in the remaining cases.

The second method makes use of the criteria for the nature of the roots of a cubic congruence as developed elsewhere by the writer.* Let p be a prime > 3, and let R be the discriminant (product of squares of differences of the roots) of a cubic

$$(5) \qquad y^3 + \beta y + b = 0.$$

The cubic congruence has a *single* integral root if, and only if, R is a quadratic non-residue of p; it has three distinct integral roots if, and only if, R is the residue of a square $81\mu^2 \neq 0$ and $e=\frac{1}{2}(-b+\mu\sqrt{-3})$ is the residue of the cube of a number $r+s\sqrt{-3}$ in which r and s are integers; it has no integral root if, and only if, R is a quadratic residue and e is not the residue of a cube.

To obtain the reduced form (5) of (1), set $z=y+\frac{2}{3}a$. Then

$$\beta=-\tfrac{1}{3}c, \qquad b=-\tfrac{1}{27}(4a-3)c, \qquad c=4a^2-6a+9,$$
$$R=-4\beta^3-27b^2=c^2.$$

We may take $\mu=\frac{1}{9}c$. Then $27e=c(2a+3\omega)$. Now

$$c=(2a+3\omega)(2a+3\omega^2).$$

If $2a$ is -3ω or $-3\omega^2$, (5) has the triple root $y=0$. For the remaining values of $2a$,

$$e=\left(\frac{2a+3\omega}{3}\right)^3 A, \qquad A=\frac{2a+3\omega^2}{2a+3\omega}\neq 0.$$

* *Bulletin of the American Mathematical Society*, vol. 13 (1906), p. 1.

Hence the problem is to determine the values of a for which A is a residue of a cube. Since $A \neq 1$, each A gives a definite value for $2a$.

First, let -3 be a quadratic residue of p. Then p is of the form $3l + 1$. Then A is the residue of a cube if, and only if, $A^l \equiv 1 \pmod{p}$, as shown by Fermat's theorem. There are $l - 1$ integral values of $A \neq 1$, and therefore $l - 1 = \frac{1}{3}(p - 4)$ values of $2a$. Incorporating the values -3ω, $-3\omega^2$, we conclude that there are in all $\frac{1}{3}(p + 2)$ values of $2a$ for which the congruence has three integral roots, including the two cases in which it has a triple root.

Next let -3 be a quadratic non-residue of p. Then p is of the form $3l - 1$. Since (4) is now irreducible modulo p, its roots $-\omega$ and $-\omega^2$ are Galois imaginaries and each is the p^{th} power of the other. To give a direct verification,

$$(-\omega)^p = -\omega^{3l-1} = -\omega^{-1}(\omega^3)^l = -\omega^2.$$

Hence $\omega^p = \omega^2$, so that $A^p = 1/A$. Thus $A^{p+1} = 1$. Now

$$(\rho + \sigma\omega)^{p^2} \equiv \rho + \sigma\omega \pmod{p},$$

when ρ and σ are integers. Hence if $A \neq 0$ is the cube of an expression $\rho + \sigma\omega$, we have

$$A^{\frac{1}{3}(p^2-1)} \equiv (\rho + \sigma\omega)^{p^2-1} \equiv 1 \pmod{p}.$$

But the greatest common divisor of $p + 1$ and $\frac{1}{3}(p^2 - 1)$ is now l. Hence the conditions are that $A^l \equiv 1$, $A \neq 1$. Thus there are $l - 1$ values of A and the same number for $2a$. The values -3ω, $-3\omega^2$ are now not integers. Hence the total number of values of $2a$ for which the congruence has three integral roots is $\frac{1}{3}(p - 2)$.

The results obtained by the two methods agree completely. The second method furnishes an explicit equation $A^l \equiv 1$ which is satisfied by the values of $2a$ yielding congruences with integral roots. To solve $A^l \equiv 1 \pmod{p}$, when $p = 3l + 1$, we have only to read off from a table of indices for the prime p the numbers A whose exponents are multiples of 3 $\pmod{p - 1}$. For the case $p = 3l - 1$, we employ similarly a table* of indices for the Galois field of order p^2.

* Bussey, *Bulletin American Math. Society*, vol. 12, p. 21, vol. 16, p. 188.

The cubics obtained by Mr. Escott are included in the wider class of cubics whose discriminant is a perfect square. For any such cubic the roots are rational functions of a single root and the cubic admits of three linear fractional transformations into itself.† In the cubic discussed here these are defined by the functions (3).

THE UNIVERSITY OF CHICAGO,
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† Serret, *Algèbre supérieure*, vol. 2, pp. 466–469.

THE HARMONICS OF A STRETCHED STRING VIBRATING IN A RESISTING MEDIUM

By C. R. Dines

We shall here consider the harmonics of a string, fastened at both ends and vibrating transversely with small amplitude in a resisting medium, the vibration being due to the tension in the string and to a force which is a linear function of the transverse velocity and the distance of each point of the string from its position of equilibrium.*

It is well known that when a string vibrates in a non-resisting medium, the component notes are in perfect harmony with the fundamental note of the string, but in the medium we are considering this is not always the case.† We shall show, however, in §2, that by a proper choice of the linear function, any two notes may be made harmonic. We shall then consider, in § §3–5, the harmonics of different relative fundamental notes for that linear function, showing that certain notes will have an infinite number of harmonics, others only a finite number, and that, for certain values of the linear function, there is one note which has no harmonics. From these results we shall, in §6, determine the curve into which the string must be initially distorted in order that only harmonics may be obtained for a certain note, and show that, in certain cases, the equation of this curve may be an infinite trigonometric series, in others a finite trigonometric series, and in certain cases the problem has no solution.

The problem reduces to one in the representation of numbers by a binary quadratic form whose type may be any of the three: parabolic, elliptic, or hyperbolic. Thus, aside from the intrinsic importance of the problem, it is of interest also because it is an application of the theory of numbers to mathematical physics. The best known problem of this kind is that of the vibrating drum-head,‡ but here the quadratic form must be elliptic, while, in the case we shall consider, the hyperbolic and parabolic types may also occur.

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* These conditions will be physically realized if we consider the string vibrating in air and tied at each point to its position of equilibrium by elastic bands.

† See Byerly, *Fourier's Series and Spherical Harmonics*, Article 65, page 115.

‡ See Riemann-Weber, *Partielle Differential-Gleichungen der mathematischen Physik*, vol. 2, p. 253. Also Byerly, *Fourier's Series and Spherical Harmonics*, art. 72, p. 129.

1. The equation of vibration in the case under discussion is obtained by adding

$$2k\frac{\partial y}{\partial t} + cy$$

to the left-hand side of the equation of vibration in a non-resisting medium with no force acting other than that due to the tension in the string.† This equation is, then

$$(1) \qquad \frac{\partial^2 y}{\partial t^2} - a^2\frac{\partial^2 y}{\partial x^2} + 2k\frac{\partial y}{\partial t} + cy = 0.$$

The boundary and initial values to be satisfied are

(a) $x = 0,\ y = 0,$ (b) $x = l,\ y = 0,$ (c) $t = 0,\ y = f(x).$

By a method similar to the usual treatment of the case where c is zero* we obtain as normal solutions satisfying the boundary condition (a)

$$y = e^{-kt}\sin ax\cos t\sqrt{a^2a^2 - (k^2 - c)},$$

$$y = e^{-kt}\sin ax\sin t\sqrt{a^2a^2 - (k^2 - c)}.$$

Thus the series giving the vibration is seen to be

$$(2)\ y = e^{-kt}\sum_{m=1}^{\infty}\left[A_m\cos t\sqrt{\frac{m^2\pi^2a^2}{l^2} - (k^2 - c)} + B_m\sin t\sqrt{\frac{m^2\pi^2a^2}{l^2} - (k^2 - c)}\right]\left[\sin\frac{m\pi x}{l}\right].$$

Let us now consider the period of any component note. It is clear that the factor e^{-kt} affects merely the amplitude, causing the vibration to die down, and does not affect the period.

Equation (2) may be written

$$(3) \qquad y = e^{-kt}\sum_{m=1}^{\infty} C_m\left[\cos\sqrt{\frac{m^2\pi^2a^2}{l^2} - (k^2 - c)}\ (t - \tau)\right]\sin\frac{m\pi x}{l},$$

where

$$A_m = C_m\cos\tau\sqrt{\frac{m^2\pi^2a^2}{l^2} - (k^2 - c)}, \qquad B_m = C_m\sin\tau\sqrt{\frac{m^2\pi^2a^2}{l^2} - (k^2 - c)},$$

* See Byerly, loc. cit. art. 1, p. 2 (VIII).
† See Byerly, loc. cit. art. 65, p. 113.

$$\tau = \sqrt{\frac{l^2}{m^2\pi^2a^2 - l^2(k^2-c)}}\tan^{-1}\frac{B_m}{A_m}, \qquad C_m^2 = A_m^2 + B_m^2.$$

Now the period T of one complete vibration must be such that

$$\cos\sqrt{\frac{m^2\pi^2a^2}{l^2} - (k^2-c)}\,(t-\tau) = \cos\sqrt{\frac{m^2\pi^2a^2}{l^2} - (k^2-c)}\,(t-\tau+T),$$

and therefore

$$\sqrt{\frac{m^2\pi^2a^2}{l^2} - (k^2-c)}\;T = 2\pi,$$

or

$$T = \frac{2\pi l}{\sqrt{m^2\pi^2a^2 - l^2(k^2-c)}}.$$

The frequency of vibration or the number of vibrations per second is, then,

$$F = \frac{1}{T} = \frac{\sqrt{m^2\pi^2a^2 - l^2(k^2-c)}}{2\pi l}. \tag{4}$$

We shall now give definitions of a few terms to be used in the discussion following. The note given by a single term of the series (3) of index m is called the component note of index m, or more briefly the note m. Absolute harmonics of a fundamental component note are component notes whose frequencies are multiples of the frequency of the fundamental. Two notes are said to be relatively harmonic when their frequencies bear a rational ratio to each other.

2. If $k^2 - c \neq 0$*, we see from (4) that the frequencies of the different component notes are not exact multiples of the frequency of the fundamental, so that we shall not in general have harmonics. We shall show, however, that by a proper choice of k and c any two notes may be made harmonic and shall then discuss the harmonics of different relative fundamental notes for these chosen values of k and c.

Let F_1 and F_2 denote the frequencies of two notes m_1 and m_2. Let

$$\frac{F_1}{F_2} = \frac{h_1}{h_2},$$

where h_1 and h_2 are relatively prime integers. Then, from (4) we have

* If $k^2 - c = 0$ it will appear from equation (7) that the problem of harmonics is the same as if the string were vibrating in a non-resisting medium.

$$(5) \qquad \frac{F_1}{F_2} = \frac{\sqrt{m_1^2\pi^2a^2 - l^2(k^2 - c)}}{\sqrt{m_2^2\pi^2a^2 - l^2(k^2 - c)}} = \frac{h_1}{h_2}.$$

Now m_1, m_2, h_1, h_2 are fixed; so that (5) can be solved for $k^2 - c$, giving

$$(6) \qquad k^2 - c = \frac{\pi^2a^2}{l^2}\left[\frac{h_1^2m_2^2 - h_2^2m_1^2}{h_1^2 - h_2^2}\right].$$

3. We shall now determine the condition to be satisfied in order that any note m_3 with frequency F_3 may be harmonic with a note m_4 with frequency F_4 for the above value of $k^2 - c$, which rendered m_1 and m_2 harmonic.

In a way similar to that by which we obtained (5), we have

$$(7) \qquad \frac{F_3}{F_4} = \frac{\sqrt{m_3^2\pi^2a^2 - l^2(k^2 - c)}}{\sqrt{m_4^2\pi^2a^2 - l^2(k^2 - c)}} = \frac{h_3}{h_4}.$$

This relation (7) will give all the harmonics of m_3 if we allow h_3 to take on all possible values and determine integral values of h_4 and m_4, such that h_3 and h_4 are relatively prime, which satisfy the relation (7).

Substituting in (7) the value of $k^2 - c$ from (6), we have

$$(8) \qquad \frac{h_3}{h_4} = \frac{\sqrt{m_3^2\pi^2a^2 - \dfrac{l^2\pi^2a^2}{l^2}\left[\dfrac{h_1^2m_2^2 - h_2^2m_1^2}{h_1^2 - h_2^2}\right]}}{\sqrt{m_4^2\pi^2a^2 - \dfrac{l^2\pi^2a^2}{l^2}\left[\dfrac{h_1^2m_2^2 - h_2^2m_1^2}{h_1^2 - h_2^2}\right]}},$$

or

$$(9) \qquad [(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2)]h_4^2 - h_3^2(h_1^2 - h_2^2)m_4^2 = h_3^2(h_2^2m_1^2 - h_1^2m_2^2).$$

4. The problem, then, reduces to one in the representation of a number by a binary quadratic form, namely that of determining for what integral values of h_4 and m_4 the number

$$h_3^2(h_2^2m_1^2 - h_1^2m_2^2)$$

is representable by the form

$$(10) \qquad [(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2)]h_4^2 - h_3^2(h_1^2 - h_2^2)m_4^2.$$

We shall first determine what values of h_3 are possible in order that the equation (9) may have integral solutions. Transposing, we have

$$h_3^2[(h_1^2 - h_2^2)m_4^2 + (h_2^2m_1^2 - h_1^2m_2^2)] = [(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2)]h_4^2.$$

Now h_3^2 is a factor of the left hand side of this equation and so must be a factor of

$$h_4^2[(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2)]$$

also. But all harmonics will be obtained by taking h_3 and h_4 relatively prime, so that h_3^2 is a factor of

$$[(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2)].$$

Therefore h_3 is a factor of σ where σ^2 is the greatest square factor in

$$[(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2)].$$

We shall, however, obtain all solutions for h_4 which are relatively prime to h_3, by substituting for h_3 the value σ, determining all solutions (h_4m_4) and reducing σ/h_4 to lowest terms. For, if $h_4 = u$, $m_4 = m_4$ are solutions of (9) in the form

$$Ah_4^2 - Bh_3^2m_4^2 = Ch_3^2 \tag{11}$$

when h_3 is given the value t, and u is prime to t, it is obvious that $h_4 = u{\cdot}s$, $h_3 = t{\cdot}s$, $m_4 = m_4$ are also solutions. But by a proper choice of s, $t{\cdot}s = \sigma$, so that $h_4 = u$, $h_3 = t$, will always be among the solutions obtained by substituting σ for h_3, solving for h_4 and m_4, and reducing σ/h_4 to lowest terms.

Since we have supposed σ^2 to be the greatest square factor in

$$[(h_1^2 - h_2^2)m_3^2 -(h_1^2m_2^2 - h_2^2m_1^2)],$$

we may set

$$[(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2)] = r\sigma^2,$$

where r contains no square factor. Then, substituting $h_3 = \sigma$, (9) becomes

$$r\sigma^2h_4^2 - \sigma^2(h_1^2 - h_2^2)m_4^2 = \sigma^2(h_2^2m_1^2 - h_1^2m_2^2),$$

or

$$rh_4^2 -(h_1^2 - h_2^2)m_4^2 = h_2^2m_1^2 - h_1^2m_2^2. \tag{12}$$

5. As is usual in problems involving binary quadratic forms, we shall discuss the form (10) under three cases, hyperbolic, elliptic, and parabolic according as the discriminant

$$\Delta = [(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2)](h_1^2 - h_2^2)h_3^2$$

is positive, negative, or zero.

Case I. $\Delta > 0$. *Form hyperbolic.*

a) $$|(h_1^2 - h_2^2)m_3^2| > |h_1^2 m_2^2 - h_2^2 m_1^2|.$$

b) $$|(h_1^2 - h_2^2)m_3^2| < |h_1^2 m_2^2 - h_2^2 m_1^2|$$

and $$\operatorname{sgn}(h_1^2 m_2^2 - h_2^2 m_1^2) \neq \operatorname{sgn}(h_1^2 - h_2^2).^*$$

Case II. $\Delta < 0$. *Form elliptic.*

$$|(h_1^2 - h_2^2)m_3^2| < |h_1^2 m_2^2 - h_2^2 m_1^2|$$

and $$\operatorname{sgn}(h_1^2 m_2^2 - h_2^2 m_1^2) = \operatorname{sgn}(h_1^2 - h_2^2).$$

Case III. $\Delta = 0$. *Form parabolic.*†

$$(h_1^2 - h_2^2)m_3^2 - (h_1^2 m_2^2 - h_2^2 m_1^2) = 0.$$

We shall now discuss the methods of solution and the number of solutions in each of these cases. It is easily verified that among these solutions will be found $m_4 = m_3$, $h_4 = \sigma$. This solution may be called the identical solution since it gives rise to no new harmonics; for if $m_4 = m_3$, we get from (7), $h_4 = h_3 = \sigma$. This solution will be discarded from the solutions of the equation (9).

Case I. $\Delta > 0$.

In the discussion of this case we shall consider two sub-cases: first, Δ a perfect square; second, Δ not a perfect square.

Subcase 1. $\Delta = e^2$.

Let $$\delta = D(r, h_1^2 - h_2^2),\ddagger$$

so that $$r = \lambda\delta, \qquad h_1^2 - h_2^2 = \mu\delta.$$

Then (12) becomes

$$\lambda h_4^2 - \mu m_4^2 = \frac{h_2^2 m_1^2 - h_1^2 m_2^2}{\delta},\S$$

and if μ contains a highest square factor ι^2 such that $\mu = \iota^2\mu'$ and μ' contains no square factor, we have

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* The notation $\operatorname{sgn}\alpha = \operatorname{sgn}\beta$ shall be understood to mean that α and β have the same sign.

† Since $h_1 \neq h_2$, $h_1^2 - h_2^2 \neq 0$.

‡ i.e. the greatest common divisor of r and $h_1^2 - h_2^2$.

§ The right hand side of this equation will be integral; for from our definition of r, $(h_1^2 - h_2^2)m_3^2 - (h_1^2 m_2^2 - h_2^2 m_1^2) \equiv 0 \pmod{\delta}$ and $h_1^2 - h_2^2 \equiv 0 \pmod{\delta}$. Therefore $h_2^2 m_1^2 - h_1^2 m_2^2 \equiv 0 \pmod{\delta}$.

$$(14) \qquad \lambda h_4^2 - \mu'(\iota\, m_4)^2 = \frac{h_2^2 m_1^2 - h_1^2 m_2^2}{\delta}.$$

Equation (14) may be written in the form

$$(15) \qquad \lambda x^2 - \mu' y^2 = \frac{h_2^2 m_1^2 - h_1^2 m_2^2}{\delta},$$

where λ and μ' are relatively prime, and neither contains a square factor. Thus if the discriminant $\Delta = \lambda\mu'$ is a perfect square, $\lambda = \mu' = \pm 1$, and if we denote the different factorizations of $(h_2^2 m_1^2 - h_1^2 m_2^2)/\delta$ by $L_i N_i$, (15) may be written

$$(16) \qquad x^2 - y^2 = L_i N_i.$$

Equating the factors of the right hand side to the factors of the left, we have

$$x + y = L_i, \qquad x - y = N_i,$$

or

$$(17) \qquad x = \frac{L_i + N_i}{2}, \qquad y = \frac{L_i - N_i}{2}.$$

To reduce the solutions of (16) to those of (12), two cases must be considered.

(A) $\lambda = +1$.

In this case $h_4 = x$, $m_4 = y/\iota$. Thus to obtain integral solutions for m_4, the congruence $y \equiv 0 \pmod{\iota}$ must be satisfied.

(B) $\lambda = -1$.

In this case $h_4 = y$, $m_4 = x/\iota$, and to obtain integral values for m_4 the congruence $x \equiv 0 \pmod{\iota}$ must be satisfied.

It is evident in both of these cases that, if $h_1^2 - h_2^2$ contains no square factor, i.e., if $\iota = 1$, the solutions of (16) are the solutions of (12).

Let us now determine the number of solutions (17). This will evidently be the same as the number of decompositions of $(h_2^2 m_1^2 - h_1^2 m_2^2)/\delta$ into the product of two factors L_i and N_i such that L_i and N_i are of the same parity.

Let $$\frac{h_2^2 m_1^2 - h_1^2 m_2^2}{\delta} = 2^\nu (a_1^\alpha \cdot a_2^\beta \cdot \cdot \cdot a_p^\kappa),$$

in which the a_k's are different odd prime numbers. Two cases will be considered.

(A) $\nu = 0$.

In this case L_i and N_i are both odd, so that any decomposition of $h_2^2m_1^2 - h_1^2m_2^2)/\delta$ will give integral solutions for x and y. As is well known, the number of different divisors of $(a_1^\alpha \cdot a_2^\beta \cdots a_\rho^\kappa)$ is

$$(\alpha + 1)(\beta + 1) \cdots (\kappa + 1) = N.$$

From this we find immediately the number of different decompositions if we consider two cases.

(a) $(h_2^2m_1^2 - h_1^2m_2^2)/\delta$ is not a perfect square.

There will in this case be just half as many decompositions as there are different divisors, or $N/2$. This will be integral since at least one of the numbers $(\alpha, \beta, \cdots \kappa)$ is odd.

(b) $(h_2^2m_1^2 - h_1^2m_2^2)/\delta = E^2$.

In this case E might be regarded as a double divisor, since $E \cdot E$ is a factorization. Thus the number of different decompositions will be $(N + 1)/2$. However, in the case $L_i = N_i = E$, the y solution is 0, which gives either $h_4 = 0$, or $m_4 = 0$. If $h_4 = 0$, then also $F_4 = 0$, which case will be discussed in §5, Case III. The solution $m_4 = 0$ is trivial, and will be discarded, hereafter, in the discussion. The number of solutions will be taken then as $(N-1)/2$.

(B) $\nu \neq 0$.

(a) $(h_2^2m_1^2 - h_1^2m_2^2)/\delta$ is not a perfect square.

Denote by C the number of decompositions obtained in (A) (a). Then with each L_i of these we can combine $2, 2^2, 2^3, \cdots 2^{\nu-1}$, the remaining even factors $2^{\nu-1}, 2^{\nu-2}, \cdots 2$, respectively, being combined with the corresponding N_i to make it of the same parity as L_i. The number of decompositions such that L_i and N_i are of the same parity is, then

$$(\nu - 1)C.$$

(b) $(h_2^2m_1^2 - h_1^2m_2^2)/\delta = E^2$.

Denote by C' the number of solutions obtained in (A) (b), above. Then, reasoning as in (B) (a), the number of solutions is seen to be

$$(\nu - 1)C' - \frac{\nu}{2}.*$$

* It is evident that ν cannot be odd, if $(h_2^2m_1^2 - h_1^2m_2^2)/\delta$ is a perfect square.

Therefore aside from the trivial solution we have, as the total number of solutions

$$(\nu - 1)\cdot C'' - \frac{\nu}{2} - 1.$$

To sum up, then, we have seen that, *the number of harmonics of the note m_3 will be finite, and an upper limit has been found for their number, this upper limit being exactly equal to the number of harmonics if $h_1^2 - h_2^2$ contains no square factor.*

Subcase 2. $\Delta \neq e^2$.

In case the discriminant of the form is not a perfect square we shall use a method of solution frequently employed in representation problems, which we briefly describe as follows:

Let the form $ax^2 + 2bxy + cy^2$ be denoted by (a, b, c) and its discriminant $b^2 - ac$ by Δ as before.

In order that m may be representable by (a, b, c) it is necessary that Δ be a quadratic residue of m, in other words that the congruence

$$n^2 \equiv \Delta (\mathrm{mod}\ m)$$

be solvable. Let n be a solution of this congruence and p be determined by the equation

$$p = \frac{n^2 - \Delta}{m}.$$

The necessary and sufficient condition that m be representable by (a, b, c) is that (m, n, p) and (a, b, c) belong to the same class. For this it is necessary and sufficient: (1) that the first roots* of the characteristic equations

$$aw^2 + 2bw + c = 0,$$

$$mw^2 + 2nw + p = 0,$$

developed into continued fractions, give rise to periods composed of the same elements in the same order, so that the period of one is given by a circular permutation of the elements of the period of the other; and (2) that the number of elements be odd, or, if even, that the number of elements which in each fraction precede the same element of the period shall be of the same parity.

* The first root of $ax^2 + 2bx + c = 0$ is defined to be $\dfrac{-b + \sqrt{b^2 - ac}}{a}$.

Thus if there is a representation we must have, in terms of continued fractions

$$w_{(a,b,c)} = (a_1, a_2, \cdots a_k, A, B \cdots L, A, B \cdots L, \cdots),$$

$$w_{(m,n,p)} = (a'_1, a'_2, \cdots a'_{k'}, A, B \cdots L, A, B \cdots L, \cdots),$$

and the number of elements in the period $(A, B, \cdots L)$ must be odd; or, if even, then $k + k'$ must be even.

Let $$x = (A, B \cdots L, A, B \cdots L, \cdots),$$

and denote the kth and $(k-1)$th convergents of $(a_1, a_2 \cdots a_k)$ by P/Q and P_1/Q_1 respectively; the k'th and $(k'-1)$th convergents of $(a'_1, a'_2 \cdots a'_{k'})$ by P'/Q' and P'_1/Q'_1 respectively; and the last and next to last convergents of the terminating continued fraction $(A, B, \cdots L)$ by p/q and p_1/q_1 respectively.

Let $\begin{pmatrix} P & P_1 \\ Q & Q_1 \end{pmatrix}$ denote the substitution

$$x = Px' + P_1y', \qquad y = Qx' + Q_1y',$$

and set

$$K = \begin{pmatrix} P & P_1 \\ Q & Q_1 \end{pmatrix}, \qquad H = \begin{pmatrix} P' & P'_1 \\ Q' & Q'_1 \end{pmatrix}, \qquad L = \begin{pmatrix} p & p_1 \\ q & q_1 \end{pmatrix}.$$

Let $dx^2 + 2ex + f = 0$ be the quadratic equation with integral coefficients of which x is a root; then, $L^n (n = \cdots -2, -1, 0, 1, 2, \cdots)$ will be automorphic substitutions of the form (d, e, f).* Now K will transform (a, b, c) into (d, e, f); H will transform (m, n, p) into (d, e, f) and therefore KL^nH^{-1} will transform (a, b, c) into (m, n, p). If

$$KL^nH^{-1} = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix},$$

then $x = \alpha_n$, $y = \gamma_n$ gives a representation of m, viz., $a\alpha_n^2 + 2b\alpha_n\gamma_n + c\gamma_n^2 = m$. The infinity of substutions KL^nH^{-1} gives an infinity of representations $x = \alpha_n$, $y = \gamma_n$.

If there is one solution, there will exist corresponding values of K, L, and H, and hence an infinite number of solutions. But we have already pointed

* Called the reduced form of (a, b, c).

out that there is one solution, viz., the identical solution $m_4 = m_3$, $h_4 = \sigma$. *The existence of this one solution, then, establishes the existence of an infinite number of solutions.*

Applying this result, we see in the present case, that, *if m_1 and m_2 are harmonic, then with every tone m_3 is associated an infinite system of harmonics.*

CASE II. $\Delta < 0$.

As in (2), Case I, we determine the form (m, n, p). In the present case, however, the reduced form, which we shall denote by (A, B, C), is such that the relation between the coefficients is

$$C > A \geqq 2|B|.$$

From the Theory of Quadratic Forms, we know that any form may be transformed into the reduced form of its class by repeated applications of the modular substitutions

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.^*$$

As in (2) of Case I, we determine the substitution KLH^{-1}, L being an automorphic substitution of (A, B, C), K the product of the transformations required to bring (a, b, c) into (A, B, C) and H^{-1} the inverse of the product of the transformations required to bring (m, n, p) into (A, B, C). But in this case there are not an infinite number of automorphic substitutions of the reduced form (A, B, C). Their number depends on the number of integral solutions (t, u) of the Pellian equation $t^2 - \Delta u^2 = \sigma^2$, where $\sigma = D(A, 2B, C)$.

In any case there are two automorphs. If $-4\Delta = 3\sigma^2$, there will be four others, while if $-4\Delta = 4\sigma^2$, there will be two others. If $b = 0$, there cannot be six automorphs of (a, b, c), and there can be four automorphs only when $a = c$. For, if $b = 0$ the coefficients of (A, B, C) will be the same as those of (a, b, c), in the same or the reverse order. There, cannot, then, be six automorphic substitutions of the reduced form, for $-4\Delta \neq 3\sigma^2$, since $4ac \neq 3[D(a, c)]^2$. There can be four automorphs only when $4ac = 4[D(a, c)]$, that is, when $a = c$.

Thus the reduced form of the class to which the form (10) under discussion belongs, will have two automorphic substitutions except when $r = h_2^2 - h_1^2$, in which case there are four. We shall consider these cases separately.

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* See Cahen, *Theorie des nombres*, p. 222.

If $r = h_2^2 - h_1^2$, (12) reduces to

$$h_4^2 + m_4^2 = \frac{h_2^2 m_1^2 - h_1^2 m_2^2}{h_2^2 - h_1^2}.$$

The right hand side of this equation will be integral, for, from our definition of r,

$$[(h_1^2 - h_2^2)m_3^2 - (h_1^2 m_2^2 - h_2^2 m_1^2)] \equiv 0 \pmod{r};$$

and if $r = h_2^2 - h_1^2$, then

$$h_2^2 m_1^2 - h_1^2 m_2^2 \equiv 0 \pmod{[h_1^2 - h_2^2]}.$$

Thus equation (12) will reduce to

$$x^2 + y^2 = k, \tag{18}$$

where $x = h_4$, $y = m_4$ and

$$k = \frac{h_2^2 m_1^2 - h_1^2 m_2^2}{h_2^2 - h_1^2}.$$

This form has been so exhaustively studied that a detailed discussion of it is unnecessary here. We shall merely state that, when k is representable by $x^2 + y^2$, there will, if μ is the number of different prime factors of k, be 2^μ representations if (x, y) and $(-x, -y)$ are regarded as the same representation, but (x, y) and (y, x) as different representations.*

One of these will be the identical solution and the other will be $h_4 = m_3$, $m_4 = h_3$, which might be called the inverse of the identical solution. But this latter solution gives rise to no new harmonics, since from (8), if $m_3 \gtreqless m_4$, then $h_3 \gtreqless h_4$. Thus if $h_4 = m_3$, $m_4 = h_3$, then $h_4 = h_3 = m_4 = m_3$. This solution is an identical solution and will be discarded. Thus we shall have in this case $2^\mu - 2$ harmonics, where μ is the number of different prime factors of

$$\frac{h_2^2 m_1^2 - h_1^2 m_2^2}{h_2^2 - h_1^2}.$$

If r is not equal to $h_2^2 - h_1^2$, a detailed discussion is necessary. Let R be the number of solutions of the congruence $n^2 \equiv \Delta \pmod{m}$. Then for each of these R solutions, there are always two automorphic substitutions giving altogether $2R$ representations. We shall consider now (x, y), $(-x, -y)$, $(-x, y)$, $(x, -y)$ as the same solution, and we shall have $\frac{1}{2}R$ as the number

* Dedekind-Dirichlet, *Vorlesungen über Zahlentheorie*, p. 68.

of solutions. Discarding the identical solution, we have $\frac{1}{2}R - 1$ as the number of harmonics. If k is prime there will be no harmonics since the congruence $x^2 \equiv a \pmod{p}$, where p is a prime, has only two solutions.

Thus we see, that, in the present case, *if* m_1 *and* m_2 *are harmonics, then with every tone* m_3 *is associated a finite number of harmonics.*

Case III. $\Delta = 0$.

In this case, we have $(h_1^2 - h_2^2)m_3^2 = h_1^2 m_2^2 - h_2^2 m_1^2$,

or
$$m_3^2 = \frac{h_1^2 m_2^2 - h_2^2 m_1^2}{h_1^2 - h_2^2}.$$

Also from (9),
$$m_4^2 = \frac{h_1^2 m_2^2 - h_2^2 m_1^2}{h_1^2 - h_2^2},$$

that is
$$m_3 = m_4,$$

and the note m_3 has no harmonics.

In fact the note m_3 has an infinite period as is seen from the equation (4). It is obvious from (3) that the vibration of the string in this case is not periodic, for the only function of t entering into the equation of vibration is e^{-kt}.

We shall outline a method for determining what values of m_3, h_1, h_2, will give us this case when m_1 and m_2 are chosen.

We have
$$\frac{h_1^2}{h_2^2} = \frac{m_1^2 - m_3^2}{m_2^2 - m_3^2}.$$

Now h_1 prime to h_2 will give us all the harmonics of m_3; thus in order that the above relation may hold, a necessary condition is

$$m_1^2 - m_3^2 = ah_1^2, \qquad m_2^2 - m_3^2 = ah_2^2 \qquad (a, \text{ integral}),$$

or
$$h_2^2 - h_1^2 = \frac{m_2^2 - m_1^2}{a}.$$

Since h_1 and h_2 are integral, a is limited in value to the factors of $m_2^2 - m_1^2$.

Let
$$\frac{m_2^2 - m_1^2}{a} = L_i N_i.$$

Then
$$h_2^2 - h_1^2 = L_i N_i,$$

and
$$h_2 = \frac{L_i + N_i}{2}, \quad h_1 = \frac{L_i - N_i}{2}.$$

If these give integral values for h_1 and h_2, we determine m_3 by the relation

$$m_3^2 = m_1^2 - ah_1^2,$$

and if m_3 is integral, the relation necessary for this case is obtained. If we consider h_1 always smaller than h_2, i.e., m_1 less than m_2, we need give a only positive values, noting that ah_1^2 cannot be greater than m_1^2. It is evident that, if m_1, m_2, m_3 have values satisfying this relation, then their multiples also are solutions for the same values of h_1 and h_2.

By the method here outlined, I have found that the only values of m_1 and m_2 such that $m_1 < m_2 < 10$, for which the form (10) is parabolic, together with the corresponding values of h_1, h_2, m_3, are :

$$m_1 = 3, \quad m_2 = 7, \quad h_1 = 1, \quad h_2 = 3, \quad m_3 = 2,$$

$$m_1 = 2, \quad m_2 = 7, \quad h_1 = 1, \quad h_2 = 4, \quad m_3 = 1.$$

We have, then, shown in §5 that: *if the form* (10) *is hyperbolic, the note m_3 may have either a finite or an infinite number of harmonics; if* (10) *is elliptic there are only a finite number of harmonics of m_3; and if* (10) *is parabolic there are no harmonics of m_3.*

6. The results that we have obtained in the preceding sections enable us to determine the curve $y = f(x)$ into which the string must be initially distorted in order to obtain a musical note, that is, so that all the component notes may be harmonics.

Putting $t = 0$ in (2), we obtain

$$f(x) = e^{-kt} \sum_{m=1}^{\infty} A_m \sin \frac{m\pi x}{l}.$$

Now, if we wish the note m_k, not harmonic, to drop out, we must have $A_k = 0$ in the equation of the initial curve.

Thus we see that, *in certain cases, we shall have as the equation of the initial curve, an infinite trigonometric series, in others, a finite trigonometric series, and in others, there is no solution for the problem.*

7. We shall now give a few examples, illustrative of the general discussion. In these we shall first choose values for h_1, h_2, m_1, m_2, thus fixing the medium in which the string vibrates. Then we shall consider the harmonics of the different notes of the string, when vibrating in that medium.

Example 1. $m_1 = 2, \quad m_2 = 3, \quad h_1 = 1, \quad h_2 = 2.$

The parabolic case does not occur.

The elliptic case occurs only for $m_3 = 1$.

Thus the form is hyperbolic for $m_3 > 1$. There is no value of m_3 for which the discriminant of the form is a perfect square.

(*a*) $m_3 = 1. \qquad h_3 = \sigma = 2.$

Equation (9) reduces to

$$h_4^2 + 3m_4^2 = 7.$$

Here we have $\Delta = -3$, and the congruence

$$n^2 \equiv -3 \pmod 7$$

has only two incongruent solutions. Thus, $R = 2$, and there is only one solution, viz., the identical solution

$$h_4 = h_3 = 2, \qquad m_4 = m_3 = 1.$$

(*b*) If $m_3 > 1$, the note m_3 has an infinite number of harmonics. We shall consider the case for $m_3 = 4$. We have here

$$m_3 = 4,$$

$$h_3 = \sigma = 1,$$

$$3m_4^2 - 41h_4^2 = 7,$$

$$\Delta = 123,$$

and the congruence,

$$n^2 \equiv 123 \pmod 7 \equiv 4 \pmod 7,$$

has the incongruent solutions $n = \pm 2$. Since a change in sign of n changes only the signs of the variables, we shall use only $n = +2$. This gives

$$p = -17,$$

$$(m, n, p) = (7, 2, -17),$$

$$w_{(3,0,-41)} = (3\,;\, 1, 2, 3, 2, 1, 6\,;\, 1, 2, 3, 2, 1, 6\,;\, \cdots)$$

$$w_{(7,2,-17)} = (1, 3, 2, 1, 6\,;\, 1, 2, 3, 2, 1, 6\,;\, 1, 2, 3, 2, 1, 6\,;\, \cdots)$$

$$K = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$$

$$H^{-1} = \begin{pmatrix} 10 & -13 \\ -67 & 87 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 221 & 33 \\ 154 & 23 \end{pmatrix}$$

$$KH^{-1} = \begin{pmatrix} -37 & 48 \\ 10 & -13 \end{pmatrix} \qquad m_4 = 34, \quad h_4 = 10.$$

$$KL_1H^{-1} = \begin{pmatrix} -4 & -7 \\ -1 & -2 \end{pmatrix} \qquad m_4 = 4, \quad h_4 = 1.$$

$$KL_1^{-1}H^{-1} = \begin{pmatrix} 9024 & 11719 \\ 2441 & 3170 \end{pmatrix} \qquad m_4 = 9024, \quad h_4 = 2441.$$

The second pair of solutions is the identical solution. From the first pair, we see that the ratio of the frequency of the thirty-fourth note to that of the fourth is $1:10$. From the third pair, we see that the ratio of the frequency of the nine-thousand-and-twenty-fourth note to that of the fourth is $1:2441$. By using the infinite number of L's, we may obtain an infinite number of harmonics.

Thus, if the second and third notes are harmonic, with the ratio of their frequencies $1:2$, the first note has no harmonics and any other note than the first has an infinite number of harmonics.

Example 2. $m_1 = 1, \quad m_2 = 5, \quad h_1 = 1, \quad h_2 = 2.$

The form (10) is hyperbolic for every value of m_3. The only value of m_3 for which Δ is a perfect square is $m_3 = 3$. The only solution of equation (9) for $m_3 = 3$ is the identical solution.

Thus, if the first and fifth notes are harmonics, with the ratio of their frequencies $1:2$, the third note has no harmonics and any note other than the third has an infinite number of harmonics.

Example 3. $m_1 = 4, \quad m_2 = 9, \quad h_1 = 2, \quad h_2 = 3.$

The form (10) is hyperbolic for every value of m_3. The only value of m_3 for which Δ is a perfect square is $m_3 = 8$, and for $m_3 = 8$ the only solution of equation (9) is the identical solution.

Thus, if the fourth and ninth notes are harmonics, with frequencies in the ratio 2 : 3, the eighth note has no harmonics and any other note has an infinite number of harmonics.

Example 4. $m_1 = 2, \quad m_2 = 3, \quad h_1 = 1, \quad h_2 = 4.$

The form (10) is elliptic for $m_3 = 1$, and hyperbolic for $m_3 > 1$. The discriminant of the form cannot be a perfect square. The only solution of equation (9) for $m_3 = 1$ is the identical solution.

Thus, if the second and third notes are harmonics, with the ratio of their frequencies 1 : 4, the first note has no harmonics and any note other than the first has an infinite number of harmonics.

Example 5. $m_1 = 2, \quad m_2 = 7, \quad h_1 = 1, \quad h_2 = 4.$

The form (10) is parabolic for $m_3 = 1$, and hyperbolic for $m_3 > 1$. The discriminant cannot be a perfect square.

Thus, if the second and seventh notes are harmonics, with the ratio of their frequencies 1 : 4, the first note has no harmonics, and any other note has an infinite number of harmonics.

Example 6. $m_1 = 1, \quad m_2 = 8, \quad h_1 = 4, \quad h_2 = 5.$

The form (10) is hyperbolic for every value of m_3, and the discriminant is a perfect square for $m_3 = 17$ or 55. The fifty-fifth note is the only harmonic of the seventeenth note and vice versa. These two notes, then, form a system of harmonics. With any other note of the string is associated an infinite system of harmonics.

Example 7. $m_1 = 3, \quad m_2 = 7, \quad h_1 = 1, \quad h_2 = 3.$

The form (10) is elliptic for $m_3 = 1$, parabolic for $m_3 = 2$, and hyperbolic for $m_3 > 2$. The discriminant of the form cannot be a perfect square. The only solution of equation (9) for $m_3 = 1$ is the identical solution.

Thus, we see that, if the third and seventh notes are harmonics with the ratio of their frequencies 1 : 3, the first and second notes have no harmonics, and any other note has an infinite number of harmonics.

THE MIXING EFFECT OF SURFACE WAVES

By Chas. S. Slichter

The following results are based upon the assumption that the "mixing effect" when a portion of a liquid is changed in shape is proportional to the magnitude of the shear. If we confine our attention to motion in two dimensions only, the problem before us is to determine to what extent particles of a liquid migrate and take up positions adjacent to particles of liquid not formerly their neighbors when a prism of liquid of square cross section changes to a prism of rectangular cross section. Theory requires that the particles of a "perfect" liquid hold the same relative position to each other after change as before. The extent to which this is not true in the case of an actual liquid constitutes what is designated as "mixing." Thus if the temperature of a cube of liquid varies uniformly from one of the faces to the opposite face, then after shear the relative distribution of temperature should be precisely the same as before shear. The liquid is "mixed" to the extent that the relative distribution of temperature has been disturbed.

If the dimensions of a rectangular parallelopiped of a liquid change from 1, 1, 1, to $1 + a$, 1, $1/(1 + a)$, the liquid is said to have experienced a finite shear of amount a. Hence, according to this definition, unit finite shear corresponds to change of cube 1, 1, 1, to parallelopiped 2, 1, 1/2. For rates of movement sufficiently low so that turbulent motion does not result, the initial assumption made above that mixing is proportional to shear seems sufficiently obvious if the present definition* of the magnitude of finite shear be adopted.

It is obvious that the amount of mixing that takes place in an element of liquid when subjected to unit shear must depend upon properties of the liquid which can be readily determined only by experiment. The viscosity, the density, and various molecular properties probably all play important parts in determining the amount of mixing that takes place when an element of a liquid changes shape.

* It should be noted that this is not the definition adopted in the usual treatment of the subject of finite shear.

The primary cause of mixing in liquids, as ordinarily observed, is due to the turbulent motion that is set up when the differential velocity from layer to layer, or between the liquid and the boundary reaches a certain critical amount. It can readily be seen by consulting the diagrams in the papers by Hele-Shaw in *Nature*, Vol. 50, page 34, and Vol. 60, page 446, that slow movements in which turbulent motions have been eliminated result in a very small amount of mixing, although the amount of shear be relatively large. In the wave motion of a liquid on whose surface simple progressive waves are maintained, the deformation or strain of any element is periodic and the effect of mixing must be cumulative even though the motion be so slow that turbulent motion is not set up. The mixing effect of the waves can be expressed as a function of the various constants that particularize the special wave motion under consideration. It is obvious that the variation of the mixing effect with the depth of the particle of liquid below the surface is the first problem to be considered.

The following notation is used in what follows:

Height of trochoidal wave	H	in	meters
Wave length	λ	"	"
Depth of water	d	"	"
Distance from the bottom to particle under consideration . .	y	"	"

The well known theory of trochoidal surface waves enables us to write down the following results:

The radius of the circular orbit of a particle at any depth is:

$$r = m\,H/2 = \tfrac{1}{2}He^{-2\pi(y-d)/\lambda}, \tag{1}$$

in which m is a proper fraction, the ratio of the radius of the orbit at level y, to the radius of the orbit of a surface particle.

The period of the wave motion in seconds is, if λ be measured in meters:

$$T \doteq \sqrt{2\pi\lambda/g} \doteq \frac{4}{5}\lambda^{\frac{1}{2}}; \qquad d \gtreqless \lambda, \tag{2}$$

the formula being correct to about the sixth decimal place. The error remains less than 5 per cent for all values of d satisfying*

$$4d > \lambda < d$$

* The approximation consists in omitting $\tanh 2\pi d/\lambda$ from the denominator of the theoretical expression for T^2.

Consider two particles of water which lie in the same vertical line when the crest and also when the trough of the wave passes immediately above them. Let the radius of the circular orbit of the higher particle be r_2 and of the lower particle be r_1 and let the distance between the centers of the two circular orbits be a, which may be taken as small as we please. Then the distances between these particles are,

at crest,	$a + r_2 - r_1$;
at trough,	$a - r_2 + r_1$.

Therefore the small rectangular parallelopiped at the crest, of edges

$$1 + (r_2 - r_1)/a, \qquad 1, \qquad 1/[1 + (r_2 - r_1)/a],$$

is first sheared to the cube

$$1, \qquad 1, \qquad 1,$$

at an intermediate stage, and then this cube is sheared at the trough to the rectangular parallelopiped

$$1 - (r_2 - r_1)/a, \qquad 1, \qquad 1/[1 + (r_2 - r_1)/a].$$

The total finite shear from crest to trough is therefore of magnitude:

$$2(r_2 - r_1)/a,$$

or, in the limit,

$$2dr/dy.$$

Therefore the "mixing effect" per second of time, or the coefficient of mixing, is

$$E = 4Adr/dy \div \frac{4}{5}\lambda^{\frac{1}{2}} = 15.6\ AH\lambda^{-\frac{3}{2}}e^{2\pi(y-d)/\lambda}, \qquad (3)$$

in which A is the mixing effect, or specific coefficient of mixing, due to unit shear. This factor must be experimentally determined for each liquid.

Assume waves of length 10 meters and of height one meter in water whose undisturbed depth is 10 meters, then the rate of mixing at level y is given by

$$E = E_0\, e^{2\pi(y-10)/10}, \qquad (4)$$

in which E_0 is the rate of mixing at the surface.

Table I shows the coefficient of mixing, or the mixing effect for one second of time, of a trochoidal progressive wave one meter high and of various lengths. The results are expressed in terms of the mixing effect due to unit finite shear.

TABLE I

Depth in meters below surface	Length of waves in meters.						
	5	10	15	20	25	30	40
0	1.310	0.495	0.296	0.175	0.125	0.125	0.062
1	0.406	0.264	0.176	0.127	0.099	0.077	0.053
2	0.121	0.138	0.115	0.092	0.077	0.062	0.045
3	0.032	0.086	0.076	0.068	0.059	0.050	0.040
4	0.009	0.048	0.048	0.049	0.046	0.040	0.034

The above results give the measure of the migration of particles away from their initial location or original companions. The numbers may be conveniently designated, as above, "coefficients of mixing," although mixing, in the common meaning of the term, does not take place unless another condition be present, namely, unless there be a quality varying in magnitude throughout the liquid which can be modified by the migrating particles. We do not speak of "mixing" a liquid by stirring it unless it has a different quality in some parts than in others. Migrating particles can produce no change in a perfectly homogeneous liquid. For example, if a liquid be of homogeneous temperature throughout, it will remain of homogeneous temperature no matter how turbulent the motion. If, however, the initial temperature be not homogeneous (say it falls continuously from the surface downward, as it actually does in summer in the water of a lake or pond) then the migration of particles discussed above will bring about a new distribution of temperature. The new distribution depends as much upon the gradients in the initial distribution as it does upon the coefficient of mixing. This problem will be considered only in its special bearing upon changes of temperature taking place in a sheet of water due to progressive waves upon its surface. We do not aim to consider

the mixing due to the turbulent waves near shore, nor to the effects of the wind in driving the warm surface water to lee shores. Professor E. A. Birge, who has given wide attention to all these matters, fully discusses the subject from the physical and practical side in another place.*

Let there be any quality or characteristic that remains constant in any horizontal layer of the liquid but varies in the vertical direction. For example, let there be a vertical distribution of temperature either continuous or discontinuous. If the temperature decrease with the depth below the surface there will be no tendency to set up convection currents, as long as the temperature is above 4°. When surface waves are set up in the liquid, the quality or characteristic pertaining to the various horizontal layers will gradually change, the distribution becoming immediately continuous across the former surfaces of discontinuity. Let the initial vertical distribution be defined by the equation,

$$t = f(y) \tag{5}$$

in which y is the depth of any point below the surface of the liquid [the same as $(y-d)$ above] and in which f represents a single valued function of y. Let the coefficient of mixing at any level be $\phi(y)$. At this level the initial gradient of t is

$$dt/dy = f'(y),$$

and after mixing it is

$$dt/dy = [1 - \phi(y)]f'(y). \tag{6}$$

Whence the distribution of the quality t after the "first mixing", or first unit of time, is given by

$$t_1 = f(y) - \int \phi(y)\, f'(y)\, dy + c.$$

Likewise the distribution of t after the nth interval is given by

$$t_n = \int [1 - \phi(y)]^n f'(y)\, dy + c. \tag{7}$$

* The results of these extensive studies of the biological, chemical, and physical phenomena of fresh water lakes is appearing as a series of Monographs published by the Wisconsin Geological Survey. The chemical phenomena are treated in Bulletin XXII. The physical studies will appear in a bulletin now in preparation.

In the important special case in which the mixing is due to waves, we may write

$$\phi(y) = ae^{-by}. \tag{8}$$

The forms of a and b have been found above. In like manner

$$t_n = \int (1 - a e^{-by})^n f'(y) dy + c, \tag{9}$$

which permits of simplification for special values of f.

Consider the case in which the temperature falls from the surface to the bottom as a linear function of y, so that $f(y) = -ky$. Then

$$t_n = -k \int (1 - a e^{-by})^n dy + c, \tag{10}$$

which, for large values of n, becomes

$$t_n = -k \int e^{-nae^{-by}} dy = -\frac{k}{b} \operatorname{li} e^{-nae^{-by}}, \tag{11}$$

in which li stands for the Integral Logarithmus.*

The constant of integration must be determined by a quadrature. Mechanical quadrature was used in obtaining the numerical results given below.

The important case leading to the development of $f(y)$ in a Fourier's Series may also be written down.

Let

$$f(y) = k_p \sin py,$$

then, for large values of n

$$t_n = k_p \sum_{a=0}^{a=\infty} (-na)^a (a^2b^2 + p^2)^{-\frac{1}{2}} (1/a!) \sin(py - \theta), \tag{12}$$

in which $\theta = \tan^{-1}(ab/p)$.

Returning to the case of equation (11), the Integral Logarithmus permits numerical results to be obtained as follows: Let $H = 1$ meter, $\lambda = 10$

* The Integral Logarithmus is defined by the integral $\int_0^x du/\log u$. See N. Nielsen, *Theorie des Integrallogarithmus und verwandter Transzendenten*, Leipzig, 1906. Since the publication (Leipzig, 1909) of Jahnke and Emde's *Funktionentafeln*, tables of this and the allied functions have become readily accessible. Table II was checked by use of the table of $Ei(-x)$ on page 21 of Jahnke and Emde's tables.

meters, $d = 10$ meters, and let n be sufficiently large so that appreciable values may be assigned to nA. From previous notation,

$$a = 15.6\ A/\lambda^{3/2} = 0.484A;$$

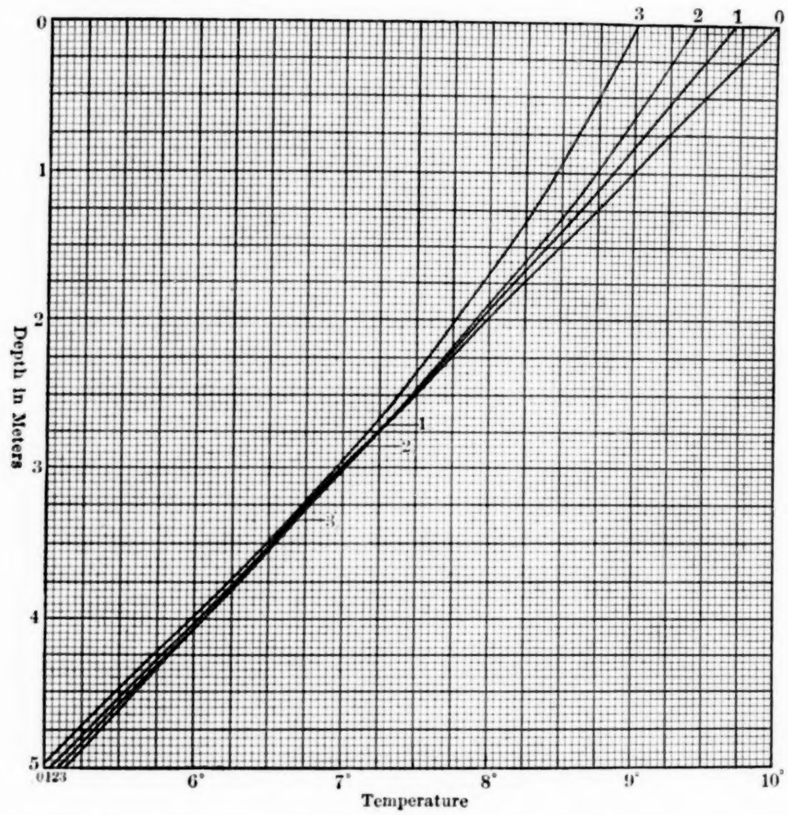
$$b = 2\pi/\lambda = 0.6283; \qquad e^{-by} = (0.534)^y.$$

As the value of A is not known, the amount of mixing after a given number of seconds cannot be found in absolute units but the amount of mixing at given *relative times* can readily be determined, since nA appears as a single variable in the results. With the above numerical data, and with nA taken equal to 1/2, 1, 2, or to elapsed times proportional to 1, 2, 4, the table II has been prepared.

Table II gives the variation in temperature in a sheet of water 10 meter deep, due to the mixing effect of surface waves of wave length 10 meters and height 1 meter, the initial temperature being equal to $(10 - y)$, where y is the depth below the surface of the water.

TABLE II

Elapsed time, nA =	0	$\frac{1}{2}$	1	2
Value of constant c =	0	0.069	0.13	0.215
Depth, y, below surface	Temp. degrees	Temp. degrees	Temp. degrees	Temp. degrees
0	10	9.70	9.45	9.06
1	9	8.87	8.75	8.48
2	8	7.96	7.92	7.78
3	7	7.01	7.01	6.98
4	6	6.04	6.07	6.09
5	5	5.05	5.11	5.15
6	4	4.06	4.11	4.18
7	3	3.06	3.12	3.19
8	2	2.07	2.12	2.20
9	1	1.07	1.12	1.21
10	0	0.07	0.13	0.22



The results are best shown graphically as in the figure. Here the original temperature is represented by the straight line 0 0. The modified temperatures at time 1/2 , 1 , 2 , 4 , are shown by the curved lines. The diagram is drawn for the upper 5 meters only. The curve for the fourth period is sketched only approximately, as the computation difficulties did not warrant greater exactness. It will be seen that the effect of the waves is gradually to change the temperature of the surface layers to a nearly uniform temperature, the absolute changes in the intervals of time being less than in earlier equal intervals. The changes in temperature show a tendency to cease rather abruptly at points where the curves cross the original gradient, as at 1_0, 2_0, 3_0. These are at depths of about 3 meters below the surface for the wave length and the wave height under consideration and gradually move downward with the lapse of time.

The same diagram may be used to show approximately the mixing effect due to waves of twice the height but of same wave length. In this case the curve 22 for wave height 2 meters corresponds to the same interval of time as the curve 11 for wave height 1 meter. The points 1, 2, 3 move to the left and the points 1_0, 2_0, 3_0, move downward at twice the rate in the case of the higher waves than in the case of the lower waves.

Although the mixing effect in the slow motion of a liquid is a magnitude of low order, yet there are important cases of motion in which the migrations of particles across the lines of flow is very great. An important case occurs in the motion of groundwater. It can be shown in this case that the movement of particles across the lines of flow equals about 30 per cent of the onward movement in the line of resultant motion. This means, for example, that impurities or contaminations are expanded or spread out over wide areas during the motion of underground waters. Diagrams showing some experimental manifestation of this fact will be found in Water Supply Paper No. 140 of the U. S. Geological Survey, pp. 22–68. The consideration of the mixing, or spread of contaminations, in groundwaters lead to the statement of a very general problem of much interest, which the writer will consider in another place.

The magnitude of the coefficient A assumed above cannot be even approximately assigned with the experimental data now at hand. It seems reasonably certain, however, that its magnitude will range higher than even the largest of the coefficients of diffusion.

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CHARACTERISTICS OF TWO PARTIAL DIFFERENTIAL EQUATIONS OF ORDER ONE*

BY C. A. NOBLE

Given two partial differential equations of order one:

$$F(x, y; z, \bar{z}, p, \bar{p}, q, \bar{q}) = 0, \tag{1}$$

$$G(x, y; z, \bar{z}, p, \bar{p}, q, \bar{q}) = 0, \tag{2}$$

where x, y are independent variables; z, $\bar{z}$ functions of these variables; and where p, $\bar{p}$, q, $\bar{q}$ denote $\partial z/\partial x$, $\partial \bar{z}/\partial x$, $\partial z/\partial y$, and $\partial \bar{z}/\partial y$, respectively. Assume that F and G are analytic in all the arguments, in the neighborhood of x_0, y_0, z_0, $\bar{z}_0$, p_0, $\bar{p}_0$, q_0, $\bar{q}_0$, where

$$F(x_0, y_0, z_0, \bar{z}_0, p_0, \bar{p}_0, q_0, \bar{q}_0) = 0,$$

$$G(x_0, y_0, z_0, \bar{z}_0, p_0, \bar{p}_0, q_0, \bar{q}_0) = 0;$$

and assume, furthermore, that $F = 0$, $G = 0$ can be written in the normal form

$$q = f(x, y; z, \bar{z}, p, \bar{p}),$$

$$\bar{q} = g(x, y; z, \bar{z}, p, \bar{p}),$$

where f and g are analytic in all their arguments.

According to the Cauchy-Kowalewski analysis, there will be two, and only two analytic functions:

$$z = \phi(x, y), \tag{3}$$

$$\bar{z} = \bar{\phi}(x, y), \tag{4}$$

which satisfy (1) and (2) simultaneously, and which satisfy also the initial conditions

$$[z]_{y=y_0} = \psi(x), \qquad [\bar{z}]_{y=y_0} = \bar{\psi}(x),$$

where ψ, $\bar{\psi}$ are arbitrary analytic functions.

---

* Read before the San Francisco Section of the American Mathematical Society, February 26, 1910.

Geometrically, this means that a unique pair of surfaces, belonging simultaneously to (1) and (2), can always be passed through the curves

$$y = y_0,\ z = \psi(x) \qquad \text{and} \qquad y = y_0,\ \bar{z} = \bar{\psi}(x).$$

Characteristics, in the present case, according to Hilbert's method of approach, would be such space curves that the Cauchy-Kowalewski analysis would not permit the inference that they determine a unique pair of surfaces of (1) and (2).

If (3), (4) are a surface pair of (1), (2), then, *in general*, the analytic curves determined upon these two surfaces by any arbitrarily selected cylinder

$$y = \lambda(x)$$

will uniquely determine the surfaces (3), (4). We seek those exceptional curves, if any exist, for which the Cauchy-Kowalewski theorem fails. In order to apply this theorem to the given equations, let us make the substitution

$$x = x,\ \eta = y - \lambda(x),$$

$$z = \phi[x,\ \eta + \lambda(x)] = \zeta(x,\ \eta),$$

$$\bar{z} = \bar{\phi}[x,\ \eta + \lambda(x)] = \bar{\zeta}(x,\ \eta).$$

In the new variables, x, η, ζ, $\bar{\zeta}$, the equations (1), (2) become

$$(1^*) \qquad F\{x,\ \eta + \lambda(x),\ \zeta,\ \bar{\zeta},\ \zeta_x - \lambda'\zeta_\eta,\ \bar{\zeta}_x - \lambda'\bar{\zeta}_\eta,\ \zeta_\eta,\ \bar{\zeta}_\eta\} = 0,$$

$$(2^*) \qquad G\{x,\ \eta + \lambda(x),\ \zeta,\ \bar{\zeta},\ \zeta_x - \lambda'\zeta_\eta,\ \bar{\zeta}_x - \lambda'\bar{\zeta}_\eta,\ \zeta_\eta,\ \bar{\zeta}_\eta\} = 0,$$

where the accents denote differentiation with respect to x.

The necessary and sufficient condition that the Cauchy-Kowalewski inference shall be invalid for the curves cut from (3) and (4) by $y = \lambda(x)$, i.e. from $\zeta = \zeta(x, \eta)$ and $\bar{\zeta} = \bar{\zeta}(x, \eta)$ by $\eta = 0$, is that (1*), (2*) shall not be solvable for ζ_η, $\bar{\zeta}_\eta$; that is, that

$$(5) \qquad \begin{vmatrix} \partial F/\partial\zeta_\eta, & \partial F/\partial\bar{\zeta}_\eta \\ \partial G/\partial\zeta_\eta, & \partial G/\partial\bar{\zeta}_\eta \end{vmatrix} = \begin{vmatrix} F_q - \lambda' F_p, & F_{\bar{q}} - \lambda' F_{\bar{p}} \\ G_q - \lambda' G_p, & G_{\bar{q}}, - \lambda' G_{\bar{p}} \end{vmatrix} = 0,$$

identically in x when for η, y, z, and $\bar{z}$ we write 0, $\lambda(x)$, $\phi[x, \lambda(x)]$, and $\bar{\phi}[x, \lambda(x)]$ respectively. If, in equation (5), we write $z = \phi(x, y)$ and

$\bar{z} = \bar{\phi}(x, y)$, but leave y undisturbed and set $\lambda' = y'$, it becomes an ordinary differential equation in x, y, and determines two one-parameter families of curves lying upon the surface $z = \phi(x, y)$ and two one-parameter families of curves lying upon $\bar{z} = \bar{\phi}(x, y)$. Whether, or not, these curves are actually curves of indetermination as to integral surfaces passing through them, remains still to be answered.

We can set up the system of ordinary differential equations for the independent determination of the characteristics, i.e. without assuming knowledge of the surfaces $z = \phi(x, y)$, $\bar{z} = \bar{\phi}(x, y)$. We have, obviously, in addition to (5),

$$(6) \qquad z' = p + qy', \qquad (7) \qquad \bar{z}' = \bar{p} + \bar{q}y',$$

$$(8) \qquad p' = r + sy', \qquad (9) \qquad \bar{p}' = r + \bar{s}y',$$

$$(10) \qquad q' = s + ty', \qquad (11) \qquad \bar{q}' = \bar{s} + \bar{t}y',$$

where the r, s, t have the usual significance.

If we differentiate (1) and (2) with respect to x regarding z, $\bar{z}$, p, $\bar{p}$, q, $\bar{q}$ as functions of x and y, eliminate r, $\bar{r}$, from the two resulting equations by the aid of (8) and (9), multiply the first by $G_q - y'G_p$, the second by $F_q - y'F_p$, and subtract, s and $\bar{s}$ will disappear by virtue of (5), and the single equation results

$$(12) \qquad (F_x + pF_z + p'F_p + \bar{p}F_{\bar{z}} + \bar{p}'F_{\bar{p}})\,(G_q - y'G_p)$$
$$- (G_x + pG_z + p'G_p + \bar{p}G_{\bar{z}} + \bar{p}'G_{\bar{p}})(F_q - y'F_p) = 0.$$

If we differentiate (1), (2) with respect to y, and proceed analogously, we obtain the further equation

$$(13) \qquad (F_y + qF_z + q'F_p + \bar{q}F_{\bar{z}} + \bar{q}'F_{\bar{p}})(G_q - y'G_p)$$
$$- (G_y + qG_z + q'G_p + \bar{q}F_{\bar{z}} + \bar{q}'G_{\bar{p}})(F_q - y'F_p) = 0.$$

Equations (1), (2), (5), (6), (7), (12), (13) suffice, in general, to determine y, z, $\bar{z}$, p, $\bar{p}$, q, $\bar{q}$ as functions of x. With initial values as indicated in the first paragraph, we should have*

* NOTE. There would be two such sets of equations as (14) and (15), since equation (5) is a quadratic in y'.

$$(14)\quad \begin{cases} y = y\{x;\ x_0,\ y_0,\ z_0,\ \bar{z}_0,\ p_0,\ \bar{p}_0,\ q_0,\ \bar{q}_0\}, \\ z = z\ \{x;\ x_0,\ y_0,\ z_0,\ \bar{z}_0,\ p_0,\ \bar{p}_0,\ q_0,\ \bar{q}_0\}, \\ p = p\{x;\ x_0,\ y_0,\ z_0,\ \bar{z}_0,\ p_0,\ \bar{p}_0,\ q_0,\ \bar{q}_0\}, \\ q = q\{x;\ x_0,\ y_0,\ z_0,\ \bar{z}_0,\ p_0,\ \bar{p}_0,\ q_0,\ \bar{q}_0\}, \end{cases}$$

$$(15)\quad \begin{cases} y = y\{x;\ x_0,\ y_0,\ z_0,\ \bar{z}_0,\ p_0,\ \bar{p}_0,\ q_0,\ \bar{q}_0\}, \\ \bar{z} = \bar{z}\{x;\ x_0,\ y_0,\ z_0,\ \bar{z}_0,\ p_0,\ \bar{p}_0,\ q_0,\ \bar{q}_0\}, \\ \bar{p} = \bar{p}\{x;\ x_0,\ y_0,\ z_0,\ \bar{z}_0,\ p_0,\ \bar{p}_0,\ q_0,\ \bar{q}_0\}, \\ \bar{q} = \bar{q}\{x;\ x_0,\ y_0,\ z_0,\ \bar{z}_0,\ p_0,\ \bar{p}_0,\ q_0,\ \bar{q}_0\}. \end{cases}$$

Equations (14) give the characteristic strip passing through (x_0, y_0, z_0) on the surface $z = \phi(x, y)$, that is, the characteristic curve, together with the tangent plane to the surface at every point of that curve. Equations (15) give the characteristic strip lying upon the surface $\bar{z} = \bar{\phi}(x, y)$ and passing through $(x_0, y_0, \bar{z}_0)$.

Consider now the two integral surfaces (3), (4) passing respectively through the curves

$$y = y_0, \quad z = \psi(x), \quad \text{and} \qquad y = y_0, \quad \bar{z} = \bar{\psi}(x).$$

Assuming that $F = 0$, $G = 0$ can be thrown into the normal form, and that the curves above selected are not characteristics, there will be no other analytic integral surface of (1), (2) through either of these curves. If we select the two elements

$$x_0,\ y_0,\ z_0 = \psi(x_0),\ q_0,\ p_0 = \psi'(x_0) \text{ and } x_0,\ y_0,\ \bar{z}_0 = \bar{\psi}(x_0),\ \bar{q}_0,\ \bar{p}_0 = \bar{\psi}'(x_0)$$

we see by (14), (15) that the characteristic strips are completely determined as lying upon their respective surfaces. If we set $x_0 = \xi$ and think of ξ as a continuous variable, these two characteristic strips will, in their motion, generate the surfaces (1), (2) respectively, upon which they lie. But $z = \psi(x)$ was selected as any analytic curve in the plane $y = y_0$ which was not a characteristic, and similarly for $z = \bar{\psi}(x)$, i.e., $\psi(\xi)$, $\bar{\psi}(\xi)$ may be thought of as arbitrary analytic functions. The equations (14), (15) then become

$$
(16)\qquad\left\{\begin{aligned}
y &= y\{x;\ \xi,\ y_0,\ \psi(\xi),\ \bar\psi(\xi),\ \psi'(\xi),\ \bar\psi'(\xi),\ q_0,\ \bar q_0\},\\
z &= z\{x;\ \xi,\ y_0,\ \psi(\xi),\ \bar\psi(\xi),\ \psi'(\xi),\ \bar\psi'(\xi),\ q_0,\ \bar q_0\},\\
p &= p\{x;\ \xi,\ y_0,\ \psi(\xi),\ \bar\psi(\xi),\ \psi'(\xi),\ \bar\psi'(\xi),\ q_0,\ \bar q_0\},\\
q &= q\{x;\ \xi,\ y_0,\ \psi(\xi),\ \bar\psi(\xi),\ \psi'(\xi),\ \bar\psi'(\xi),\ q_0,\ \bar q_0\},
\end{aligned}\right.
$$

$$
(17)\qquad\left\{\begin{aligned}
y &= y\{x;\ \xi,\ y_0,\ \psi(\xi),\ \bar\psi(\xi),\ \psi'(\xi),\ \bar\psi'(\xi),\ q_0,\ \bar q_0\},\\
\bar z &= \bar z\{x;\ \xi,\ y_0,\ \psi(\xi),\ \bar\psi(\xi),\ \psi'(\xi),\ \bar\psi'(\xi),\ q_0,\ \bar q_0\},\\
\bar p &= \bar p\{x;\ \xi,\ y_0,\ \psi(\xi),\ \bar\psi(\xi),\ \psi'(\xi),\ \bar\psi'(\xi),\ q_0,\ \bar q_0\},\\
\bar q &= \bar q\{x;\ \xi,\ y_0,\ \psi(\xi)\ \bar\psi(\xi),\ \psi'(\xi),\ \bar\psi'(\xi),\ q_0,\ \bar q_0\}.
\end{aligned}\right.
$$

The first two equations in (16) and in (17) may be looked upon as furnishing the general solution of (1) and (2), expressed in terms of the parameter ξ. The solution involves two arbitrary functions, as one should expect.

The entire foregoing process is an extension of that presented by Hilbert in his lectures in 1900–1901 (see also Hedrick, *Annals of Mathematics*, July, 1903). Here, as there, it appears that the equations of the characteristics suffice to solve completely the proposed partial differential equations. In other words, two partial differential equations of the first order with two unknown functions may be made to depend completely upon a system of ordinary differential equations.

In order to exhibit the characteristics of (1), (2) in relation to the calculus of variations, let us consider the problem of rendering the integral

$$\int_{u_0}^{u_1}\{\lambda(u)(z' - px' - qy') + \mu(u)(\bar z' - \bar px' - \bar qy')\}\,du$$

an extremum, with the auxiliary conditions

$$F = 0, \qquad G = 0,$$

whereby λ, μ are arbitrary functions, and accents denote differentiation with respect to u. The necessary conditions for the solution of the above problem, the so-called Lagrange equations, proceed from the vanishing of the first variation of the integral

$$\int_{u_0}^{u_1}\{\lambda(u)(z' - px' - qy') + \mu(u)(\bar z' - \bar px' - \bar qy') + \xi F + \eta G\}\,du,$$

where ξ, η are undetermined functions of u. These equations of condition, twelve in number, are

$$
\begin{array}{llrll}
(1) & d(\lambda p + \mu \bar{p})/du + \xi F_x + \eta G_x = 0 & (2) & d(\lambda q + \mu \bar{q})/du + \xi F_y + \eta G_y = 0 \\
(3) & \lambda' - \xi F_z - \eta G_z \qquad = 0 & (4) & \mu' - \xi F_{\bar{z}} - \eta G_{\bar{z}} \qquad = 0 \\
(5) & \lambda x' - \xi F_p - \eta G_p \qquad = 0 & (6) & \mu x' - \xi F_{\bar{p}} - \eta G_{\bar{p}} \qquad = 0 \\
(7) & \lambda y' - \xi F_q - \eta G_q \qquad = 0 & (8) & \mu y' - \xi F_{\bar{q}} - \eta G_{\bar{q}} \qquad = 0 \\
(9) & z' - px' - qy' \qquad = 0 & (10) & \bar{z}' - \bar{p}x' - \bar{q}y' \qquad = 0 \\
(11) & F \qquad = 0 & (12) & G \qquad = 0
\end{array}
$$

By means of (3), (4), (5), (6) we can eliminate λ', μ', λ, μ from (1), (2), (7), (8). The resulting equations are

$$
(13) \quad (y'/x')^2(F_pG_{\bar{p}} - F_pG_p) - y'/x'(F_pG_{\bar{q}} - F_{\bar{p}}G_q + F_qG_{\bar{p}} - F_qG_p) + F_qG_{\bar{q}} - F_qG_q = 0
$$

$$
(14),(15) \quad \frac{F_x + pF_z + p'/x'F_p + \bar{p}F_{\bar{z}} + \bar{p}'/x'F_p}{G_x + pG_z + p'/x'G_p + \bar{p}G_{\bar{z}} + \bar{p}'/x'G_p}
= \frac{F_y + qF_z + q'/x'F_p + \bar{q}F_{\bar{z}} + \bar{q}'/x'F_p}{G_y + qG_z + q'/x'G_p + \bar{q}G_{\bar{z}} + \bar{q}'/x'G_{\bar{p}}} = \frac{y'/x'F_p - F_q}{y'/x'G_p - G_q}
$$

The seven equations (9), (10), $\cdots$ (14), (15) suffice to determine the seven ratios $x' : y' : z' : \bar{z}' : p' : \bar{p}' : q' : \bar{q}'$; that is, to determine y, z, $\bar{z}$, p, $\bar{p}$, q, $\bar{q}$ as functions of x. They are identical with the seven equations obtained by the first method for determining the characteristics.

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THE DIFFERENTIAL EQUATION OF THE THIRD ORDER WITH A QUADRATIC RELATION BETWEEN THE INTEGRALS

By S. Epsteen

Let

$$T(y) \equiv \frac{d^3y}{dx^3} + t_1(x)\frac{d^2y}{dx^2} + t_2(x)\frac{dy}{dx} + t_3(x)y = 0$$

be a linear homogeneous differential equation with rational coefficients, and

$$y_1, \quad y_2, \quad y_3$$

a fundamental system of integrals between which there is a quadratic relation

$$(1) \qquad y_2^2 - y_1y_3 = 0.$$

Fuchs has shown* that there exists a linear homogeneous differential equation of second order, with rational coefficients,

$$S(z) \equiv \frac{d^2z}{dx^2} + s_1(x)\frac{dz}{dx} + s_2(x)z = 0,$$

such that

$$y_1 = z_1^2, \quad y_2 = z_1z_2, \quad y_3 = z_2^2,$$

the functions z_1 and z_2 forming a fundamental system of $S = 0$.

Vessiot† has considered the equation of third order $T(y) = 0$ with the relation between the integrals

$$(2) \qquad y_2^2 - y_1y_3 = r(x),$$

where $r(x)$ is a rational function of x.

It will be shown in this note that *by the transformation* (4) *the equation* $T(y) = 0$ *can be transformed to an equation of third order* $\overline{T}(\bar{y}) = 0$ *such that*

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* L. Fuchs, *Acta Mathematica*, vol. 1, 1882, pp. 321–362; Picard, *Traité d'Analyse*, vol. 3, (1908), pp. 585–588.

† E. Vessiot, *Annales de l'École Normale Supérieure*, vol. 9, (1892), pp. 274–180.

(3) $$\bar{y}_2^2 - \bar{y}_1\bar{y}_3 = 0.$$

This proves that the discussion of $\bar{T} = 0$ with the relation (2) between its integrals is equivalent to Fuchs' (seemingly more special) problem of discussing an equation of third order with the relation (1) between its integrals.

The equation

(4) $$\bar{y} = p_0 y + p_1 y' + p_2 y'',$$

the coefficients p_0, p_1, p_2 denoting (for the present undetermined) functions of x, can, in general, be solved for y. By differentiating twice, and making use of the fact that y is an integral of $T = 0$, we obtain

(4′) $$\bar{y}' = (p_0' - p_2 t_3)y + (p_0 + p_1' - p_2 t_2)y' + (p_1 + p_2' - p_2 t_1)y'',$$

(4″) $$\begin{aligned}\bar{y}'' = {} & [(p_0' - p_2 t_3)' - t_3(p_1 + p_2' - p_2 t_1)]y \\ & + [(p_0' - p_2 t_3) + (p_0 + p_1' - p_2 t_2)' - t_2(p_1 + p_2' - p_2 t_1)]y' \\ & + [(p_0 + p_1' - p_2 t_2) + (p_1 + p_2' - p_2 t_1)' - t_1(p_1 + p_2' - p_2 t_1)]y''.\end{aligned}$$

The solution gives

(5) $$y = q_0\bar{y} + q_1\bar{y}' + q_2\bar{y}'',$$

where q_0, q_1, q_2 are functions of x, provided we exclude values of p_0, p_1, p_2 (and consequently of q_0, q_1, q_2), called *singular*, which lead to a vanishing determinant of the coefficients on the right hand side of (4), (4′), (4″).

It is well known* that $\bar{y}$ also satisfies a linear homogeneous differential equation of third order

$$\bar{T}(\bar{y}) \equiv \frac{d^3\bar{y}}{dx^3} + \bar{t}_1(x)\frac{d^2\bar{y}}{dx^2} + \bar{t}_2(x)\frac{d\bar{y}}{dx} + \bar{t}_3(x)\bar{y} = 0.$$

We proceed to show that there exists a fundamental system $\bar{y}_1, \bar{y}_2, \bar{y}_3$ of $\bar{T} = 0$ such that

(3) $$\bar{y}_2^2 - \bar{y}_1\bar{y}_3 = 0,$$

while between the integrals of $T = 0$ the relation is

---

* L. Schlesinger, *Handbuch der Theorie der linearen Differentialgleichungen*, vol. 2, part 1, p. 114.

(2) $$y_2^2 - y_1y_3 = r(x).$$

Assuming the equality (3) we have from (4)

$$(p_0y_2 + p_1y_2' + p_2y_2'')^2 - (p_0y_1 + p_1y_1' + p_2y_1'')(p_0y_3 + p_1y_3' + p_2y_3'') = 0,$$

or, collecting,

(6) $$p_0^2(y_2^2 - y_1y_3) + p_1^2(y_2'^2 - y_1'y_3') + p_2^2(y_2''^2 - y_1''y_3'') + p_0p_1(2y_2y_2' - y_1y_3' - y_1'y_3)$$
$$+ p_0p_2(2y_2y_2'' - y_1y_3'' - y_1''y_3) + p_1p_2(2y_2'y_2'' - y_1'y_3'' - y_1''y_3') = 0.$$

By differentiating the equation (2) five times and eliminating derivatives beyond the second by means of $T(y) = 0$, we obtain five relations with rational coefficients which are linear in the following five functions:—

$$y_2'^2 - y_1'y_3' = r_1(x), \qquad y_2''^2 - y_1''y_3'' = r_2(x), \qquad 2y_2y_2' - (y_1y_3' + y_1'y_3) = r_3(x)$$
$$2y_2y_2'' - (y_1y_3'' + y_1''y_3) = r_4(x), \qquad 2y_2'y_2'' - (y_1'y_3'' + y_1''y_3') = r_5(x);$$

whence it appears that these five functions are rational in x.

Substituting these values in (6), there results

(7) $$p_0^2r + p_1^2r_1 + p_2^2r_2 + p_0p_1r_3 + p_0p_2r_4 + p_1p_2r_5 = 0.$$

In general, we may let

$$p_1 = 0, \qquad p_2 = 1$$

and obtain the equation

(7′) $$rp_0^2 + r_4p_0 + r_2 = 0$$

for the determination of p_0. Thus the transformation

(4′) $$\bar{y} = p_0y + y''$$

takes $T(y) = 0$ into $\overline{T}(\bar{y}) = 0$ and $y_2^2 - y_1y_3 = r(x)$ into $\bar{y}_2^2 - \bar{y}_1\bar{y}_3 = 0$. In case it happens that

$$p_0 = \frac{-r_4 \pm \sqrt{r_4^2 - 4rr_2}}{2r}, \qquad p_1 = 0, \qquad p_2 = 1$$

are singular values of the p's, then some other arbitrary choice of p_1 and p_2 together with the value of p_0 computed from (7) should be taken.*

* The domain of rationality of this investigation consists of: (1) all complex numbers, (2) the coefficients t_1, t_2, t_3 of the equation $T = 0$, the function $r(x)$, and (3) the operations of addition, subtraction, multiplication, division (exclusive of the null divisor), extraction of square roots, and differentiation.

It remains still to be shown that $\overline{y}_1, \overline{y}_2, \overline{y}_3$ form a fundamental system of $\overline{T}(\overline{y}) = 0$, where

$$(5') \qquad y_i = q_0\overline{y}_i + q_1\overline{y}_i' + q_2\overline{y}_i'' \qquad (i = 1, 2, 3).$$

If these functions are not linearly independent, we will have

$$c_1\overline{y}_1 + c_2\overline{y}_2 + c_3\overline{y}_3 = 0;$$

from this equation and (5′) it follows that

$$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0,$$

a condition which is impossible, since y_1, y_2, y_3 are a fundamental system of $T(y) = 0$.

Note. The propriety of replacing the equation $T(y) = 0$ by $\overline{T}(\overline{y}) = 0$ may be made clear from at least two points of view. I. If $\overline{T} = 0$ is integrated then, by (5) a fundamental system of $T = 0$ is known at once; and inversely, if $T = 0$ is integrated, then a fundamental system of $\overline{T} = 0$ is known by (4). II. From the standpoint of the Picard-Vessiot Theory, the equations $T = 0$ and $\overline{T} = 0$, being cogredient,* have the same Rationality Group, and consequently, the same integration theory.

THE UNIVERSITY OF COLORADO,
MARCH, 1911.

* Schlesinger, loc. cit. pp. 115 and 121.

RELATIONS AMONG SOME CYCLOTOMIC CUBICS

BY T. HAYASHI

1. According to Mr. E. B. Escott's valuable memoir "Cubic congruences with three real roots", published in the *Annals of Mathematics*, second series, vol. 2, pp. 86–92, January, 1910, we have the theorem:

The congruence

$$x^3 + ax^2 + bx + c \equiv 0 \qquad (\text{mod } p)$$

has three roots (when it has any), the relations between the roots being

$$\left.\begin{aligned} \beta &= \alpha^2 + k\alpha + l, \\ \gamma &= \beta^2 + k\beta + l, \\ \alpha &= \gamma^2 + k\gamma + l, \end{aligned}\right\} \tag{1}$$

where

$$b = -(a^2 - 4ak - 2a + 3k^2 + 3k + 3),$$

$$c = -(a^3 - 4a^2k - 2a^2 + 5ak^2 + 5ak + 3a - 2k^3 - 3k^2 - 3k - 1),$$

$$l = -(a^2 - 3ak - a + 2k^2 + 2k + 2).$$

This theorem can be applied to the cubic equation

$$x^3 + ax^2 + bx + c = 0 \tag{2}$$

having three real roots. As Mr. Escott remarks, the cubic equation

$$x^3 + a'x^2 + b'x + c' = 0 \tag{3}$$

which is obtained when we replace a by $-a + 3k + 1$, has three real roots among which the relations (1) exist.

The two cubics (2) and (3) may be said to be *conjugate* with respect to the relations (1).

Using this fact, I will show that the cyclotomic cubics* in the cases of the circle-division into seven, thirteen, and nineteen parts are all conjugate to that in the case of the division into nine parts.

2. The cyclotomic cubic

$$x^3 + x^2 - 2x - 1 = 0$$

whose roots are

* I will use this name for any cubic which can be used for the circle-division.

$$2\cos\frac{2\pi}{7},\qquad 2\cos\frac{4\pi}{7},\qquad 2\cos\frac{6\pi}{7},$$

is obtained by putting $a = 1$, $k = 0$.

If we replace a by $-a + 3k + 1$, i.e. 0, and therefore if we put $a = 0$, $k = 0$, then we get the cubic equation

$$x^3 - 3x + 1 = 0.$$

This cubic is the cyclotomic equation in the case of the circle-division into nine parts, because its roots are

$$2\cos\frac{2\pi}{9},\qquad 2\cos\frac{4\pi}{9},\qquad 2\cos\frac{8\pi}{9}.$$

Therefore *the cyclotomic cubic* $x^3 + x^2 - 2x - 1 = 0$ *in the case of the circle-division into seven parts is conjugate to the cyclotomic cubic* $x^3 - 3x + 1 = 0$ *in the case of the circle-division into nine parts with respect to*

$$\left.\begin{aligned}\beta &= a^2 - 2,\\ \gamma &= \beta^2 - 2,\\ a &= \gamma^2 - 2.\end{aligned}\right\}$$

If
$$a^3 + a^2 - 2a - 1 = 0,$$
$$\beta = a^2 - 2 = -\frac{1}{a+1}.$$

Similarly
$$\gamma = \beta^2 - 2 = -\frac{1}{\beta+1},$$
$$a = \gamma^2 - 2 = -\frac{1}{\gamma+1}.$$

Again if
$$a^3 - 3a + 1 = 0,$$
$$\beta = a^2 - 2 = \frac{a-1}{a}.$$

Changing the signs of a and β, and transforming,

$$a = -\frac{1}{\beta+1}.$$

Hence if a, β, γ be the roots

$$-2\cos\frac{2\pi}{9},\qquad -2\cos\frac{4\pi}{9},\qquad -2\cos\frac{8\pi}{9}$$

of the cubic equation

$$x^3 - 3x - 1 = 0,$$

then
$$\alpha = -\frac{1}{\beta+1},$$
$$\beta = -\frac{1}{\gamma+1},$$
$$\gamma = -\frac{1}{\alpha+1}.$$

Therefore by changing the order of the roots, we know that *the cyclotomic cubic $x^3 + x^2 - 2x - 1 = 0$ in the case of the circle-division into seven parts is conjugate to the cyclotomic cubic $x^3 - 3x - 1 = 0$ in the case of the division into nine parts with respect to*

$$\left.\begin{aligned} \beta &= -\frac{1}{\alpha+1}, \\ \gamma &= -\frac{1}{\beta+1}, \\ \alpha &= -\frac{1}{\gamma+1}. \end{aligned}\right\}$$

3. The cyclotomic cubic

$$x^3 + x^2 - 4x + 1 = 0$$

whose roots are

$$2\cos\frac{2\pi}{13} + 2\cos\frac{10\pi}{13}, \quad 2\cos\frac{4\pi}{13} + 2\cos\frac{6\pi}{13}, \quad 2\cos\frac{8\pi}{13} + 2\cos\frac{12\pi}{13},$$

is obtained by putting $a = 1$, $k = 1$.

If we replace a by $-a + 3k + 1$, i.e. 3, and therefore if we put $a = 3$, $k = 1$, we get the cubic equation

$$x^3 + 3x^2 - 3 = 0$$

whose roots are

$$-2\cos\frac{2\pi}{9} - 1, \qquad -2\cos\frac{4\pi}{9} - 1, \qquad -2\cos\frac{8\pi}{9} - 1.$$

Therefore *the cyclotomic cubic $x^3 + x^2 - 4x + 1 = 0$ in the case of the circle-division into thirteen parts is conjugate to the cyclotomic cubic $x^3 + 3x^2 - 3 = 0$ in the case of the circle-division into nine parts with respect to*

$$\left.\begin{aligned} \beta &= \alpha^2 + \alpha - 3, \\ \gamma &= \beta^2 + \beta - 3, \\ \alpha &= \gamma^2 + \gamma - 3. \end{aligned}\right\}$$

We can also show that *the cyclotomic cubic* $x^3 + x^2 - 4x + 1 = 0$ *in the case of the circle-division into thirteen parts is conjugate to the cyclotomic cubic* $x^3 - 3x + 1 = 0$ *in the case of the circle-division into nine parts with respect to*

$$\left.\begin{aligned} \beta &= \frac{a-1}{a}, \\ \gamma &= \frac{\beta-1}{\beta}, \\ a &= \frac{\gamma-1}{\gamma}. \end{aligned}\right\}$$

4. The cyclotomic cubic in the case of the circle-division into nineteen parts is

$$x^3 + x^2 - 6x - 7 = 0.$$

(Gauss, *Werke*, vol. 1, § 353.) This is obtained by putting $a = 1$, $k = -1$.

If we replace a by $-a + 3k + 1$, i.e. -3, and therefore if we put $a = -3$, $k = -1$, we get the cubic equation

$$x^3 - 3x^2 - 6x + 17 = 0.$$

Solving this cubic by Trigonometry, we get as its roots

$$2\sqrt{3}\cos\frac{5\pi}{18} + 1, \qquad 2\sqrt{3}\cos\frac{7\pi}{18} + 1, \qquad 2\sqrt{3}\cos\frac{17\pi}{18} + 1,$$

i.e.

$$2\cos\frac{\pi}{9} + 2\cos\frac{4\pi}{9} + 1,\ 2\cos\frac{2\pi}{9} + 2\cos\frac{5\pi}{9} + 1,\ 2\cos\frac{7\pi}{9} + 2\cos\frac{10\pi}{9} + 1;$$

so that this cubic may be said to be the cyclotomic cubic in the case of the division into nine parts.

Therefore *the cyclotomic cubic* $x^3 + x^2 - 6x - 7 = 0$ *in the case of the circle-division into nineteen parts is conjugate also to the cyclotomic cubic* $x^3 - 3x^2 - 6x + 17 = 0$ *in the case of the division into nine parts with respect to*

$$\left.\begin{aligned} \beta &= a^2 - a - 5, \\ \gamma &= \beta^2 - \beta - 5, \\ a &= \gamma^2 - \gamma - 5. \end{aligned}\right\}$$

TOKIO, JAPAN.

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